COMPLEX ANALYSIS

Lecture notes for
MA 445/545

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$$f'(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(u)}{(u-z)^2}\, du$$

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Contents

Preface

This is a (hopefully gentle) introduction to the beautiful field of Complex Analysis also called the Theory of Functions.

Students will work on the material (Exercises and Theorems) and present their findings in class. Exceptions are Theorems marked with a ✓ or a $\sharp$. The former (✓) are considered known from Calculus or their proofs are very similar to those of analogous theorems in Calculus. The latter ($\sharp$) will be accepted without proof, their proofs are provided in graduate courses.

CHAPTER 1

The complex numbers

1.1. The algebra of complex numbers

DEFINITION 1. The field of complex numbers, denoted by $\mathbb{C}$, is obtained when two operations, an addition and a multiplication, are introduced in $\mathbb{R}^2$, the set of all ordered pairs of real numbers. The addition is

$$(a, b) + (c, d) = (a + c, b + d)$$

and the multiplication is

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

where a, b, c, d are arbitrary real numbers.

The real numbers a and b, which make up the complex number (a, b), are called its *real part* and *imaginary part*, respectively. We use the notation $\operatorname{Re} z$ and $\operatorname{Im} z$ to denote the real and imaginary parts of the complex number z. Note that even the imaginary part of a complex number is a real number.

EXERCISE 1. Choose two complex numbers of your liking. Then add them and multiply them. Identify real and imaginary parts of sum and product.

THEOREM 1. Addition is *associative*, i.e., if a, b, c, d, f and g are real numbers so that $r = (a, b)$, $s = (c, d)$, and $t = (f, g)$ are complex numbers, then

$$(r + s) + t = r + (s + t).$$

Consequently, we may write unambiguously $r + s + t$.

THEOREM 2. Addition is *commutative*, i.e., if a, b, c and d are real numbers so that $r = (a, b)$ and $s = (c, d)$ are complex numbers, then

$$r + s = s + r.$$

THEOREM 3. The complex number $(0, 0)$ is a *neutral element of addition*, i.e., for all $(a, b) \in \mathbb{C}$ we have

$$(a, b) + (0, 0) = (a, b).$$

No other complex number is a neutral element of addition.

THEOREM 4. If (a, b) is a complex number then the complex number, $(-a, -b)$ is a *negative* of (a, b), i.e.,

$$(a, b) + (-a, -b) = (0, 0).$$

No other complex number can serve as a negative of (a, b).

THEOREM 5. Multiplication is *associative*, i.e., if a, b, c, d, f and g are real numbers so that $r = (a, b)$, $s = (c, d)$, and $t = (f, g)$ are complex numbers, then

$$(rs)t = r(st).$$

Consequently, we may write unambiguously rst.

THEOREM 6. Multiplication is *commutative*, i.e., if a, b, c and d are real numbers so that $r = (a, b)$ and $s = (c, d)$ are complex numbers, then

$$rs = sr.$$

THEOREM 7. The complex number $(1, 0)$ is a *neutral element of multiplication*, i.e., for all $(a, b) \in \mathbb{C}$ we have

$$(a, b)(1, 0) = (a, b).$$

No other complex number is a neutral element of multiplication.

EXERCISE 2. Is there a complex number s such that $rs = (1, 0)$ when $r = (3, \pi)$?

THEOREM 8. Every complex number $r \neq (0, 0)$ has a unique *reciprocal*, i.e., there is one and only one complex number s such that $rs = (1, 0)$.

THEOREM 9. Multiplication is distributive over addition, i.e., if a, b, c, d, f and g are real numbers and $r = (a, b)$, $s = (c, d)$, and $t = (f, g)$ are complex numbers, then

$$r(s + t) = rs + rt.$$

NOTATION 1. Let z and w be complex numbers and assume $w \neq (0, 0)$. We write $-z$ for the negative of z and $w^{-1} = 1/w$ for the reciprocal of w. Instead of $z(1/w)$ we write z/w.

DEFINITION 2. The complex number $(0, 1)$ is called the *imaginary unit* and is denote by i.

THEOREM 10. Let a and b be real numbers. Then

$$(a, 0) + \mathrm{i}(b, 0) = (a, b).$$

THEOREM 11. We have $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0)(b, 0) = (ab, 0)$.

NOTATION 2. Theorem 11 shows that complex numbers whose imaginary part is 0 behave just like real numbers under addition and multiplication. Therefore we will, from now on, simply write a in place of $(a, 0)$. Thus we consider the set of real numbers to be a subset of the set of complex numbers.

Using Theorem 10 we can (and will) write $a + \mathrm{i}b$ in place of (a, b). The numbers $\mathrm{i}b$, when b is a real number, are called *purely imaginary* numbers.

THEOREM 12. Suppose z and w are complex numbers. We have $zw = 0$, if and only if $z = 0$ or $w = 0$ (or both).

THEOREM 13. $\mathrm{i}^2 = -1$.

EXERCISE 3. Find the real and imaginary parts of the following numbers: (i) $\sqrt{3} - 2\mathrm{i} + \mathrm{i}(4 + \mathrm{i}3\sqrt{2})$ and (ii) $(-3 + \mathrm{i})(2\mathrm{i} + 5)$.

DEFINITION 3. If $a, b \in \mathbb{R}$ and $z = a + \mathrm{i}b$, then $\overline{z} = a - \mathrm{i}b$ is called the *complex conjugate* of z.

THEOREM 14. Let z and w be complex numbers. Then the following statements are true:

(1) $\operatorname{Re} z = (z + \overline{z})/2$.
(2) $\operatorname{Im} z = (z - \overline{z})/(2\mathrm{i})$.
(3) $\overline{\overline{z}} = z$.
(4) $\overline{z + w} = \overline{z} + \overline{w}$.

(5) $\overline{zw} = \overline{z}\,\overline{w}$.

THEOREM 15. $z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 > 0$ unless $z = 0$.

DEFINITION 4. The non-negative real number $|z| = \sqrt{z\overline{z}}$ is called the *absolute value* or *modulus* of z.

THEOREM 16. Let z and w be complex numbers. Then the following statements are true:

(1) $|z| = 0$ if and only if $z = 0$.
(2) $|z| = |\overline{z}|$.
(3) $|zw| = |z||w|$.
(4) $|\operatorname{Re} z| \le |z|$.
(5) $|\operatorname{Im} z| \le |z|$.

THEOREM 17. For any two complex numbers z and w the triangle inequality

$$|z + w| \le |z| + |w|$$

holds.

Hint: Note that $z\overline{w} + \overline{z}w = 2\operatorname{Re}(z\overline{w}) \le 2|z||w|$.

The next result follows immediately from the triangle inequality and is sometimes also referred to by that name.

THEOREM 18. The following inequalities hold for any two complex numbers u and v.

$$|u| - |v| \le |u + v| \quad \text{as well as} \quad |v| - |u| \le |u + v|.$$

Both inequalities can be combined as

$$\big||u| - |v|\big| \le |u + v|.$$

1.2. The geometry and topology of complex numbers

DEFINITION 5. The non-negative number $|z - w|$ is called the *distance* between the complex numbers z and w.

THEOREM 19. Like any distance function the distance between complex numbers has the following properties.

(1) $|z - w| = 0$ if and only if $z = w$.
(2) $|z - w| = |w - z|$.
(3) $|z - w| \le |z - u| + |u - w|$ (triangle inequality).

Here z, w, u are arbitrary complex numbers.

You may want to convince yourself that the distance between the complex numbers $a + \mathrm{i}b$ and $c + \mathrm{i}d$ is the same as the distance between the points (a, b) and (c, d) in $\mathbb{R}^2$ which is defined as the magnitude of the vector $(c - a, d - b)$. Hence, as metric spaces, $\mathbb{R}^2$ and $\mathbb{C}$ are the same. The set $\mathbb{C}$ is therefore often called the *complex plane*.

DEFINITION 6. The *unit circle* is the set of all those complex numbers whose distance from 0 is equal to one, i.e., those complex numbers z satisfying $|z| = 1$.

THEOREM 20. If z is a point on the unit circle, then there is a unique number $\theta \in (-\pi, \pi]$ such that $z = \cos\theta + \mathrm{i}\sin\theta$and any such point is on the unit circle. If z is any non-zero complex number, then $z/|z|$ is on the unit circle. Hence, if $z \ne 0$, then there is a unique positive number r and a unique number $\theta \in (-\pi, \pi]$ such that $z = r(\cos\theta + \mathrm{i}\sin\theta)$.

Definition 7. The numbers r and θ defined in Theorem 20 are called the *polar coordinates* of z.

Exercise 4. Find the polar coordinates of the complex numbers $1+\mathrm{i}$, $-3-3\mathrm{i}$, and $\sqrt{3}-3\mathrm{i}$.

Definition 8. Let $z_0 \in \mathbb{C}$ and $r \geq 0$. The set $D(z_0, r) = \{z : |z - z_0| < r\}$ is called the *open disk* of radius r centered at z_0. The set $\overline{D}(z_0, r) = \{z : |z - z_0| \leq r\}$ is called the *closed disk* of radius r centered at z_0.

Definition 9. A subset U of the complex plane is called *open*, if, for any point $z \in U$, there is an open disk D such that $z \in D \subset U$. A subset C of the complex plane is called *closed*, if its complement is open.

Theorem 21. An open disk is open and a closed disk is closed.

Definition 10. Let S be subset of $\mathbb{C}$. The complex number z is a *limit point* of S, if, for all $n \in \mathbb{N}$, the set $(S \setminus \{z\}) \cap D(z, 1/n)$ is not empty.

Exercise 5. Is 0 a limit point of $S = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z = 0\}$?

Definition 11. The set $\{tw + (1-t)z : t \in [0,1]\}$ is called the *line segment* joining the complex numbers z and w. A subset S of the complex plane is called *convex*, if, together with any two points z and w in S, the line segment joining z and w is also in S.

Exercise 6. Show the line segment joining 1 and i is in $D(1+\mathrm{i}, 2)$.

Theorem 22. Any disk, open or closed, is convex.

CHAPTER 2

Differentiation

2.1. Limits and continuity

Definition 12. Let S be subset of $\mathbb{C}$ and $f : S \to \mathbb{C}$ a function from S to $\mathbb{C}$. Suppose that z_0 is a limit point of S. We say that f *converges* to the complex number L as z tends to z_0, if the following statement is true:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall z \in S : 0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon.$$

The number L is called a limit of f at z_0. We write $L = \lim_{z \to z_0} f(z)$ tacitly assuming that $z \in S$.

Exercise 7. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = 2z - 3\mathrm{i}$.

(1) If the function f converges as z tends to $z_0 = 1 + 2\mathrm{i}$, guess what it would converge to. Call that number L.
(2) Let $\varepsilon = 1$. Find δ such that $|f(z) - L| < 1$ whenever $|z - (1 + 2\mathrm{i})| < \delta$.
(3) Do the same for $\varepsilon = 1/100$ and $\varepsilon = 10^{-6}$.
(4) Find a friend and play the following game: A gives B a positive number called ε, B finds a positive number δ which has the property defined earlier. B wins, if she can always find δ, regardless which number ε A gives her. A wins if B cannot find δ. If B wins the game we have that f converges to L as z tends to z_0.

Theorem 23. If it exists, the limit of f at z_0 is unique.

Hint: $|L_1 - L_2| = |L_1 - f(z) + f(z) - L_2| \le |f(z) - L_1| + |f(z) - L_2|$.

Theorem 24. Let S be subset of $\mathbb{C}$ and z_0 a limit point of S. The function $f : S \to \mathbb{C}$ has a limit at z_0 if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ have limits at z_0.

Definition 13. Let S be subset of $\mathbb{C}$ and $f : S \to \mathbb{C}$ a function from S to $\mathbb{C}$. Suppose $z_0 \in S$ is a limit point of S. We say that f is *continuous* at z_0, if f converges to $f(z_0)$ as z tends to z_0. We also call f continuous at z_0 if $z_0 \in S$ is not a limit point of S. We say that f is *continuous*, if f is continuous at every point in S.

Theorem 25 (✓). Let S be subset of $\mathbb{C}$ and $f : S \to \mathbb{C}$ a function from S to $\mathbb{C}$. The function f is continuous at $z_0 \in S$, if and only if the following statement holds:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall z \in S : |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

Theorem 26 (✓). Let S be subset of $\mathbb{C}$, α a complex number, and f, g functions from S to $\mathbb{C}$ which are continuous at the point $z_0 \in S$. Then αf, $f + g$, and fg are also continuous at z_0.

Theorem 27 (✓). Suppose S_1 and S_2 are subsets of $\mathbb{C}$, that f is a function from S_1 to S_2, and that g is a function from S_2 to $\mathbb{C}$. If f is continuous at $z_0 \in S_1$ and if g is continuous at $f(z_0) \in S_2$, then $f \circ g$ is continuous at z_0.

EXERCISE 8. Show that the function $z \mapsto 3z^2 - 5z + 2$, defined on all of $\mathbb{C}$, is continuous using Theorem 26.

EXERCISE 9. Show that the reciprocal function $z \mapsto 1/z$, defined on $\mathbb{C} \setminus \{0\}$, is continuous.

EXERCISE 10. Show that the function $z \mapsto 1/(3z+5)$, defined on $\mathbb{C} \setminus \{-5/3\}$ is continuous using Theorem 27.

2.2. Holomorphic functions

DEFINITION 14. Let S be a subset of $\mathbb{C}$, $f : S \to \mathbb{C}$ a function, and $z_0 \in S$ a limit point of S. We say that f is *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the *derivative* of f at z_0 and is commonly denoted by $f'(z_0)$. If $f : S \to \mathbb{C}$ is differentiable at every point $z_0 \in S$, we say that f is *differentiable* on S.

This definition is completely analogous to that of a derivative of a function of a real variable. In fact, when S is a real interval (one of the cases we are interested in) things are hardly different from Real Analysis; the fact that f assumes complex values is not very important. However, if S is an open set and z may approach z_0 from many directions, the existence of the limit has far-reaching consequences.

EXERCISE 11. Show that the power function $z \mapsto z^2$ is everywhere differentiable and find the derivative.

THEOREM 28 (✓). Suppose S, S_1 and S_2 are subsets of $\mathbb{C}$. Then the following statements hold true:

(1) A function $f : S \to \mathbb{C}$ is differentiable at a point $z_0 \in S$ if and only if there is a number F and a function $h : S \to \mathbb{C}$ which is continuous at z_0, vanishes there, and satisfies $f(z) = f(z_0) + D(z - z_0) + h(z)(z - z_0)$. In this case $D = f'(z_0)$.
(2) If $f : S \to \mathbb{C}$ is differentiable at $z_0 \in S$, then it is continuous at z_0.
(3) Sums and products of differentiable functions with common domains are again differentiable.
(4) The chain rule holds, i.e., if $f : S_1 \to S_2$ is differentiable at z_0 and $g : S_2 \to \mathbb{C}$ is differentiable at $f(z_0)$, then $g \circ f$ is differentiable at z_0, and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

EXERCISE 12. If $f(z) = 1/(z + 1)$ and $z_0 = 1$ find a number D and a function h as in item (1) of the previous theorem.

THEOREM 29. Suppose S is a real interval. Then the function $f : S \to \mathbb{C}$ is differentiable at $a \in S$ if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentiable there. In this case,

$$f'(a) = (\operatorname{Re} f)'(a) + i(\operatorname{Im} f)'(a).$$

HINT: Use item (1) of Theorem 28.

The situation is completely different if S is an open set in $\mathbb{C}$.

EXERCISE 13. Show that $z \mapsto \operatorname{Re}(z^2)$ is *not* differentiable at z_0 if $z_0 \neq 0$ (the same is true for $z \mapsto \operatorname{Im}(z^2)$). Hint: Approach the point z_0 along different straight lines.

DEFINITION 15. Let Ω be a non-empty open set. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic* on Ω if it is differentiable at every point of Ω. A function which is defined and holomorphic on all of $\mathbb{C}$ is called *entire*.

It is easy to think that the notions of differentiability and holomorphicity are the same. To avoid this mistake note that differentiability is a pointwise concept while holomorphicity is defined on open sets. Nevertheless, we may say that a function is holomorphic at a point, if it is holomorphic on some disk centered at that point.

THEOREM 30 (✓). Holomorphic functions are continuous on their domain. Sums, differences, and products of holomorphic functions (on common domains) are holomorphic. The composition of holomorphic functions is also holomorphic. The usual formulas hold, including the chain rule.

EXERCISE 14. Show that the functions $z \mapsto 3z^2 - z + 5$ and $z \mapsto (z + 3\mathrm{i})/(z - \mathrm{i})$ are holomorphic on their domains (identify the domains).

CHAPTER 3

Integration

3.1. Integrals

Definition 16. Let $[a,b]$ be a closed interval in $\mathbb{R}$ and f a complex-valued function on $[a,b]$. Then $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions on $[a,b]$. We say that f is *(Riemann) integrable* over $[a,b]$ if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are. The integral is defined to be

$$\int_a^b f = \int_a^b \operatorname{Re} f + \mathrm{i} \int_a^b \operatorname{Im} f.$$

Exercise 15. Let $f : [0,\pi] \to \mathbb{C}$ be defined by $f(x) = \cos(x) + \mathrm{i}\sin(x)$. Compute $\int_0^\pi f$.

Theorem 31. Suppose f and g are functions which are integrable over $[a,b]$ and α is a complex number. Then

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b \alpha f = \alpha \int_a^b f.$$

We say that the integral is linear.

Theorem 32. If f is integrable over $[a,b]$, then

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Hint: To prove this inequality let $z = \int_a^b f$ and, when this is different from 0, set $\alpha = |z|/z$. Then consider $\int_a^b \alpha f$ which is non-negative.

3.2. Paths

Definition 17. Let $S \subset \mathbb{C}$ be non-empty and $[a,b]$ a non-trivial bounded interval in $\mathbb{R}$. A continuous function $\gamma : [a,b] \to S$ is then called a *path* in S. If the derivative γ' exists and is continuous on $[a,b]$, γ is called a *smooth path*.

The set $\{\gamma(t) : t \in [a,b]\}$, called the *image* of γ, is denoted by γ^*.

The points $\gamma(a)$ and $\gamma(b)$ in Ω are called *initial point* and *end point* of γ, respectively. A path is called *closed* if its initial and end points coincide.

Definition 18. The number $\int_a^b |\gamma'(t)| dt$ is called the *length* of the smooth path γ.

Exercise 16. Compute the lengths of the paths $\gamma_1 : [0, 2\pi] \to \mathbb{C} : t \mapsto \cos(t) + \mathrm{i}\sin(t)$ and $\gamma_2 : [0,1] \to \mathbb{C} : t \mapsto 3 + 4t - \mathrm{i}(2 - 3t)$.

Definition 19. Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path in $\mathbb{C}$ and $f : \gamma^* \to \mathbb{C}$ is continuous. Then the number

$$\int_a^b (f \circ \gamma)\gamma'$$

is well-defined. It is called the *integral* of f along γ and denoted by $\int_\gamma f$.

EXERCISE 17. Compute the integral of f along the smooth path γ in the following situations:

(1) $\gamma : [0,1] \to \mathbb{C} : t \mapsto t^2 - \mathrm{i}t$ and $f : \mathbb{C} \to \mathbb{C} : z \mapsto z^2$.
(2) $\gamma : [0,1] \to \mathbb{C} : t \mapsto e^t + \mathrm{i}t^2$ and $f : \mathbb{C} \to \mathbb{C} : z \mapsto 2\,\mathrm{Re}(z) + 2\,\mathrm{Im}(z)$.

THEOREM 33. Suppose z_0 is a fixed point in $\mathbb{C}$ and $r > 0$ is a fixed number. Let the closed smooth path γ be defined by $\gamma : [0, 2\pi] \to \mathbb{C} : t \mapsto z_0 + r(\cos(t) + \mathrm{i}\sin(t))$ and let $f : \mathbb{C} \setminus \{z_0\} \to \mathbb{C} : z \mapsto 1/(z - z_0)$. Then

$$\int_\gamma f = 2\pi\mathrm{i}.$$

THEOREM 34. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a smooth path and f is a continuous function on Ω^*. Then we have the estimate

$$\left|\int_\gamma f\right| \leq \max\big(|f(\gamma(t))| : t \in [a, b]\big)\, L(\gamma)$$

where $L(\gamma)$ is the length of γ.

3.3. Contours

THEOREM 35. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a smooth path. Then $\eta : [0, 1] \to \mathbb{C} : t \mapsto \eta(t) = \gamma((1 - t)a + tb)$ is also a smooth path with the same range and the same initial and end points as γ. Moreover, $\int_\eta f = \int_\gamma f$.

THEOREM 36. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a smooth path. Then $\eta : [a, b] \to \mathbb{C} : t \mapsto \eta(t) = \gamma(a + b - t)$ is also a smooth path with the same range as γ. However, the initial and end points of γ and η are switched. Moreover, $\int_\eta f = -\int_\gamma f$.

DEFINITION 20. Let γ and η be as in Theorem 36. Then η is called the *opposite* of γ. We use the notation $\eta = \ominus\gamma$.

NOTATION 3. Suppose we have two smooths paths γ_1 and γ_2. Then we denote the (ordered) pair (γ_1, γ_2) by $\gamma_1 \oplus \gamma_2$. Instead of $\gamma_1 \oplus (\ominus\gamma_2)$ we will simply write $\gamma_1 \ominus \gamma_2$. Of course, we can extend this notation to any finite number of smooths paths, e.g., $\gamma_1 \oplus ... \oplus \gamma_n = \bigoplus_{k=1}^n \gamma_k$.

DEFINITION 21. Suppose $\gamma_1, ..., \gamma_n$ are smooth paths in Ω. Then $\Gamma = \bigoplus_{k=1}^n \gamma_k$ is called a *contour* in Ω. The *image* Γ^* of the contour Γ is the set $\bigcup_{k=1}^n \gamma_k^*$.

DEFINITION 22. If $\Gamma = \bigoplus_{k=1}^n \gamma_k$ is a contour in S and f a continuous function on Γ^*, we define the integral of f along Γ by

$$\int_\Gamma f = \sum_{k=1}^n \int_{\gamma_k} f.$$

EXERCISE 18. Let γ_k, $k = 1, ..., 5$ be the paths respectively defined on $[0, 1]$ by $\gamma_1(t) = \cos(2\pi t) + \mathrm{i}\sin(2\pi t)$, $\gamma_2(t) = t$, $\gamma_3(t) = 1 + \mathrm{i} - t$, $\gamma_4(t) = 1 + \mathrm{i}t$, and $\gamma_5(t) = \mathrm{i} - \mathrm{i}t$. Let $\Gamma = \gamma_1 \oplus ... \oplus \gamma_5$. If $f(z) = z^2$ find $\int_\Gamma f$.

DEFINITION 23. A contour $\gamma_1 \oplus ... \oplus \gamma_n$ is called *closed* when there is a permutation π of $\{1, ..., n\}$ such that the end point of γ_k coincides with the initial point of $\gamma_{\pi(k)}$ for $k = 1, ..., n$.

EXERCISE 19. Determine whether the contour Γ from Exercise 18 is closed.

Definition 24. A particularly important instance of a contour $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$ is when for $k = 1, ..., n-1$, the end point of γ_k coincides with the initial point of γ_{k+1} (perhaps after reordering the indices). A contour of this type is called a *connected contour*. The initial point of γ_1 is called the *initial point* of Γ while the end point of γ_n is called the *end point* of Γ. We say a contour connects x to y, if it is a connected contour with initial point x and end point y.

Exercise 20. Let γ_2, ..., γ_4 be as in Exercise 18 and define $\Gamma = \gamma_2 \oplus ... \oplus \gamma_4$. Is Γ a connected contour?

Notation 4. Let $z_1, ..., z_{n+1} \in \mathbb{C}$. We define $\gamma_k(t) = (1-t)z_k + tz_{k+1}$ for $t \in [0, 1]$ and $k = 1, ..., n$, and $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$. We use the notation $\Gamma = \langle z_1, ..., z_{n+1} \rangle$.

Theorem 37. The γ_k defined in Notation 4 are smooth paths with initial point z_k and end point z_{k+1}. γ_k^* is the line segment joining z_k and z_{k+1}. The length of the line segment equals the length of γ_k.

Definition 25. A subset S of $\mathbb{C}$ is called connected, if, for any two points in S there is a connected contour with image in S joining the points.

Theorem 38. Convex sets are connected.

Definition 26. Let S be a subset of $\mathbb{C}$. A component of S is the set of all points in S which are connected to a given point by a connected contour with image in S.

Theorem 39. Suppose S be a subset of $\mathbb{C}$. Let $C(z)$ be the set of all points in S which are connected to z. Then $w \in C(z)$ if and only if $z \in C(w)$.

Exercise 21. How many connected components does $\mathbb{C} \setminus \{0\}$ have? Is it connected?

Exercise 22. How many connected components does $\mathbb{C} \setminus \{z : |z| = 1\}$ have? Is it connected?

Theorem 40. If S is open then so are its connected components.

Hint: If $C(z)$ is one of the components of S and $w \in C(z)$, show that $D(w, r) \subset C(z)$ for some $r > 0$.

3.4. Primitives

Definition 27. Suppose Ω is a non-empty open set and $F' = f$ on Ω. Then F is called a *primitive* of f.

Theorem 41. If F is a primitive of f and c is a complex number, then $F + c$ is also a primitive of f.

Theorem 42. If $\gamma : [a, b] \to \Omega$ is a smooth path and the continuous function f has a primitive F in Ω, then $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$.

Hint: Use the chain rule.

Theorem 43. The primitives of the zero function in a non-empty connected open set are precisely the constant functions.

Hint: If F is a primitive of the zero function, consider $F(y) - F(x) = \int_\Gamma F'$ whenever Γ is a contour connecting x to y.

CHAPTER 4

Analytic functions

4.1. Analytic functions

Definition 28. Let a_n, $n \in \mathbb{N}_0$, be a sequence of complex numbers and z_0 a fixed complex number. Then the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, $z \in \mathbb{C}$, is called a *power series*. The series is called convergent for z, if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n(z-z_0)^n$ exists and it is called absolutely convergent for z, if $\lim_{N\to\infty} \sum_{n=0}^{N} |a_n||z-z_0|^n$ exists.

Recall the ratio test from Calculus whose proof also works for series with complex terms.

Theorem 44 (✓). Consider the series $\sum_{n=0}^{\infty} a_n$ and assume that $L = \lim_{n\to\infty} |a_{n+1}/a_n|$ exists. If L is strictly less than 1, then the series is (absolutely) convergent. If it is strictly larger than 1, then the series is divergent.

Theorem 45 (✓). A series converges, if it converges absolutely.

Theorem 46 (✓). Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ there are the following three possibilities:

(1) The series converges absolutely for every $z \in \mathbb{C}$.
(2) The series converges only for $z = z_0$.
(3) There is a positive number R such that the series converges absolutely whenever $|z-z_0| < R$ and diverges whenever $|z-z_0| > R$.

Definition 29. The number R in Theorem 46 is called the *radius of convergence* and the disk $D(z_0, R)$ is called the *disk of convergence*. If the first case holds we say that the radius of convergence is infinite and if the second case holds we say that the radius of convergence is 0.

Exercise 23. Find the radius of convergence and the limit of the power series $f(z) = \sum_{n=0}^{\infty} z^n$. Hint: First find the value of the partial sums $s_N(z) = \sum_{n=0}^{N} z^n$ for $N = 0, 1, 2, 3$, then multiply with $1 = (1-z)/(1-z)$ and simplify the numerator; do you see a pattern?

Definition 30. We say that a function $f : D(z_0, R) \to \mathbb{C}$ is represented by a power series if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for all $z \in D(z_0, R)$.

Definition 31. Let Ω be a non-empty open set. A function $f : \Omega \to \mathbb{C}$ is called *analytic* in Ω, if Ω is a union of open disks in each of which f is represented by a power series.

Exercise 24. The function $z \mapsto 1/(1-z)$ is analytic in $D(0, 1)$, the open unit disk.

Exercise 25. The function $z \mapsto 1/(1-z)$ is analytic in $D(\mathrm{i}, \sqrt{2})$. You may use that

$$\frac{1}{1-z} = \frac{1}{1-\mathrm{i}} \frac{1}{1-\frac{z-\mathrm{i}}{1-\mathrm{i}}}.$$

Exercise 26. The function $z \mapsto 1/(1-z)$ is analytic in $\mathbb{C} \setminus \{1\}$.

THEOREM 47 (✓). The series $\sum_{n=0}^{\infty} n^k a_n(z-z_0)^n$, $k \in \mathbb{N}_0$, have the same radius of convergence regardless of k.

If we differentiate the series $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ term by term we get $z \mapsto \sum_{n=1}^{\infty} n a_n z^{n-1}$ and, according to the previous theorem, these two series have the same radius of convergence. As you might suspect, the latter is indeed the derivative of the former.

THEOREM 48 (✓). If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0| < R$, then $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ in the same disk, i.e., the power series may be differentiated term by term. Consequently, f is holomorphic on $D(z_0, R)$.

THEOREM 49. Every analytic function is holomorphic.

The central and most astonishing theorem of complex analysis is that the converse is also true, i.e., every holomorphic function is analytic, but we will have to wait until Theorem 65 to see this.

THEOREM 50 (Taylor series). Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0| < R$, i.e., $f : D(z_0, R) \to \mathbb{C}$ is analytic. Then f is infinitely often differentiable and $a_n = f^{(n)}(z_0)/n!$. We call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

the Taylor series of f about z_0.

THEOREM 51 (♯). Suppose ψ and ϕ are continuous complex-valued functions on $[a, b]$. Then the function f defined by

$$f(z) = \int_a^b \frac{\psi(t)}{\phi(t) - z} dt$$

is analytic in $\Omega = \mathbb{C} \setminus \phi([a, b])$. In fact, if $z_0 \in \Omega$, if $r = \inf\{|\phi(t) - z_0| : t \in [a, b]\}$, and if $z \in D(z_0, r)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where

$$a_n = \int_a^b \frac{\psi(t)dt}{(\phi(t) - z_0)^{n+1}}.$$

4.2. The exponential function

THEOREM 52. The series $\sum_{n=0}^{\infty} z^n/n!$ converges for every $z \in \mathbb{C}$.

DEFINITION 32. The function defined by the series in Theorem 52 is called the *exponential function* and is denoted by exp. In other words,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

THEOREM 53. The exponential function has the following properties:

(1) If $t \in \mathbb{R}$, then $\exp(\mathrm{i}t) = \cos(t) + \mathrm{i}\sin(t)$, in particular, $\exp(0) = 1$.
(2) $\exp'(z) = \exp(z)$, in particular, exp is entire.
(3) $\exp(a)^{-1} = \exp(-a)$.
(4) $\exp(a + b) = \exp(a)\exp(b)$.
(5) The exponential function has no zeros.

(6) The exponential function is periodic with period 2πi.
(7) Any period of the exponential function is an integer multiple of $2\pi i$. In particular, $\exp(z) = 1$ if and only if z is an integer multiple of $2\pi i$.

HINTS: For (1) recall the power series for sin and cos, see, e.g., Stewart, Essential Calculus, Section 8.7. For (3) find the derivative of the function f defined by $f(z) = \exp(z)\exp(-z)$. Similarly, for (4) use $f(z) = \exp(z+b)\exp(-z)$.

4.3. The index of a point with respect to a closed contour

THEOREM 54. If $\gamma : [a,b] \to \mathbb{C}$ is a smooth path and $z \in \mathbb{C} \setminus \gamma^*$, define $F : [a,b] \to \mathbb{C}$ by

$$F(s) = \exp\left(\int_a^s \frac{\gamma'(t)}{\gamma(t)-z}dt\right).$$

Then $s \mapsto F(s)/(\gamma(s)-z)$ is constant. If γ is closed, then $F(b) = F(a) = 1$.

HINT: A function defined on a real interval is constant if and only if its derivative is 0.

THEOREM 55. Let γ be a closed smooth path in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \gamma^*$. The function

$$z \mapsto \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{du}{u-z}$$

on Ω is an analytic function assuming only integer values. It is constant on each connected component of Ω and, in particular, zero near infinity.

HINTS: Use Theorem 51 to show analyticity. Define F as in 54. Then $F(b) = 1$ and, using part (7) of Theorem 53, the values of Ind must be in $\mathbb{Z}$. Why can they not change when z changes only little? What happens when z becomes very large?

THEOREM 56 (♯). Let Γ be a closed contour in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \Gamma^*$. The function Ind_Γ defined by

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{du}{u-z}$$

on Ω is an analytic function assuming only integer values. It is constant on each connected component of Ω and, in particular, zero near infinity.

EXERCISE 27. Consider the smooth paths $\gamma_1(t) = \exp(it)$, $\gamma_2(t) = \exp(it)/2$, and $\gamma_3(t) = \exp(-it)/2$ all defined on $[0, 2\pi]$ and the contours $\Gamma_1 = \gamma_1$, $\Gamma_2 = \gamma_1 \oplus \gamma_2$, and $\Gamma_3 = \gamma_1 \oplus \gamma_3$. Show that the contours are closed and find $\mathrm{Ind}_{\Gamma_k}(z)$ for $k = 1, 2,$ and 3 and for $z = 0$, $z = 3i/4$, and $z = -2$.

THEOREM 57. Let m be an integer and suppose γ is defined by $\gamma(t) = z_0 + r\exp(\mathrm{i}mt)$, $t \in [0, 2\pi]$. Then $\mathrm{Ind}_\gamma(z) = m$ for all $z \in D(z_0, r)$.

CHAPTER 5

Cauchy's theorem and some of its consequences

5.1. Cauchy's theorem

THEOREM 58 (Cauchy's theorem for functions with primitives). Suppose Ω is a non-empty open set and f a continuous function on Ω which has a primitive F. If Γ is a closed contour in Ω, then $\int_\Gamma f = 0$.

HINT: As a warm-up you may want to assume first that $n = 1$ or $n = 2$.

THEOREM 59. Suppose $n \in \mathbb{Z}$. Unless $n = -1$ the power function $z \mapsto z^n$ has the primitive $z \mapsto z^{n+1}/(n+1)$ with domain $\Omega = \mathbb{C} \setminus \{0\}$ if n is negative and $\Omega = \mathbb{C}$ if n is non-negative.

THEOREM 60 (Cauchy's theorem for integer powers). Suppose $n \in \mathbb{Z} \setminus \{-1\}$. If Γ is a closed contour in Ω, then $\int_\Gamma z^n dz = 0$.

The exceptional case $n = -1$ gives rise to interesting complications which we will discuss later. Recall, though, from Exercise 17 that $\int_\gamma dz/z = 2\pi\mathrm{i} \neq 0$ when $\gamma(t) = \cos t + \mathrm{i} \sin t$ for $t \in [0, 2\pi]$.

THEOREM 61 ($\sharp$ Cauchy's theorem for triangles). Suppose Ω is a non-empty open set, z_0 a point in Ω, and $f : \Omega \to \mathbb{C}$ a continuous function which is holomorphic on $\Omega \setminus \{z_0\}$. If Δ is a solid triangle in Ω with vertices a, b, and c, let $\Gamma = \langle a, b, c, a\rangle$. Then $\int_\Gamma f = 0$.

THEOREM 62 (Cauchy's theorem for convex sets). Suppose Ω is a non-empty open convex set, z_0 a point in Ω, and $f : \Omega \to \mathbb{C}$ a continuous function which is holomorphic on $\Omega \setminus \{z_0\}$. Fix $a \in \Omega$ and define $F : \Omega \to \mathbb{C}$ by $F(z) = \int_{\langle a,z\rangle} f$. Then F is a primitive of f and $\int_\Gamma f = 0$ for every closed contour Γ in Ω.

HINT: Note that

$$F(z+h) - F(z) = \int_{\langle z,z+h\rangle} f = \int_0^1 f(z+th)h dt = f(z)h + \int_0^1 (f(z+th) - f(z))h dt.$$

Recall that $|f(z+th) - f(z)|$ is as small as we like as long as h is sufficiently small.

5.2. Consequences of Cauchy's theorem

THEOREM 63. Suppose Ω is a non-empty open convex set, $f : \Omega \to \mathbb{C}$ a holomorphic function on Ω, z_0 is a point in Ω, and Γ is a closed contour in Ω. Let

$$g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}.$$

Then $\int_\Gamma g = 0$.

THEOREM 64 (Cauchy's integral formula). Suppose Ω and f are as in Theorem 63. If $z_0 \in \Omega \setminus \Gamma^*$, then

$$f(z_0)\operatorname{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz.$$

EXERCISE 28. Let $\gamma(t) = 2\exp(it)$, $t \in [0, 2\pi]$. Compute

(1) $\int_\gamma \exp(z)/(2z - \pi)\,dz$.
(2) $\int_\gamma z^2/(z^2 + 2z - 3)\,dz$.

We can now prove the central theorem of Complex Analysis.

THEOREM 65. A holomorphic function is analytic on its domain of definition.

HINT: Use Theorem 64 with a suitable contour Γ and Theorem 51.
From now on the words holomorphic and analytic may be considered synonymous (many authors do not ever make a distinction between them). In particular, a holomorphic function may be expanded into a Taylor series about any point in its domain.

EXERCISE 29. Show that $z \mapsto (z+3)/(z-\mathrm{i})$ is holomorphic in $\mathbb{C}\setminus\{\mathrm{i}\}$ and find its power series about the point $z_0 = 2\mathrm{i} - 1$.

THEOREM 66. Suppose Ω is a non-empty open convex set, $f : \Omega \to \mathbb{C}$ is a holomorphic function on Ω, $\gamma : [0, 1] \to \Omega$ is a closed smooth path in Ω, and z_0 is a point in $\Omega \setminus \gamma^*$. Then

$$f^{(n)}(z_0)\operatorname{Ind}_\gamma(z_0) = \frac{n!}{2\pi \mathrm{i}}\int_\Gamma \frac{f(u)}{(u - z_0)^{n+1}}du$$

for all $n \in \mathbb{N}_0$

HINT: Choose r so that $D(z_0, r)$ does not intersect γ^*. In $D(z_0, r)$ define $g(z) = f(z)\operatorname{Ind}_\gamma(z)$ and recall that $\operatorname{Ind}_\gamma(z)$ is constant there (Theorem 56). By Theorem 64

$$g(z) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f(u)}{u - z}\,du = \frac{1}{2\pi \mathrm{i}}\int_0^1 \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}\,du.$$

Now use Theorem 51 to find the Taylor coefficients of g when expanded about z_0. Finally, use the relationship between the Taylor coefficients and the values of $g^{(n)}(z_0)$ (Theorem 50).

THEOREM 67 ($\sharp$ General integral formulas for convex sets). Suppose Ω is a non-empty open convex set, $f : \Omega \to \mathbb{C}$ is a holomorphic function on Ω, Γ is a closed contour in Ω, and z_0 is a point in $\Omega \setminus \Gamma^*$. Then

$$f^{(n)}(z_0)\operatorname{Ind}_\Gamma(z_0) = \frac{n!}{2\pi \mathrm{i}}\int_\Gamma \frac{f(u)}{(u - z_0)^{n+1}}du$$

for all $n \in \mathbb{N}_0$

EXERCISE 30. Let $\gamma(t) = 2\exp(it)$, $t \in [0, 2\pi]$. Compute

(1) $\int_\gamma \exp(2z)/(2z - \pi)^3\,dz$.
(2) $\int_\gamma z^2/(z^3 + z^2 - 5z + 3)\,dz$.

THEOREM 68. Suppose $\sum_{n=0}^\infty a_n(z - z_0)^n$ is the Taylor series of the holomorphic function $f : D(z_0, R) \to \mathbb{C}$. Then its radius of convergence is at least equal to R.

HINT: Given $z \in D(z_0, R)$ there is an r such that $|z - z_0| < r < R$. Then consider the closed path $\gamma(t) = z_0 + r\exp(it)$, $t \in [0, 2\pi]$ and use Theorem 51.

THEOREM 69 (Cauchy's estimate). Suppose $f : D(z_0, R) \to \mathbb{C}$ is a holomorphic function. If $|f(z)| \leq M$ whenever $z \in D(z_0, R)$, then

$$|a_n| = \frac{|f^{(n)}(z_0)|}{n!} \leq \frac{M}{R^n}$$

for all $n \in \mathbb{N}_0$.

THEOREM 70 (Liouville's theorem). Every bounded entire function is constant.

HINT: Show that the Taylor coefficients a_1, a_2, ... must all be 0.

DEFINITION 33. If f is holomorphic and $f(z_0) = 0$, then z_0 is called a *zero* of f.

DEFINITION 34. Suppose f is a holomorphic function in $D(z_0, r)$ and $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is its Taylor series about z_0. If, for some $m \in \mathbb{N}$, we have $a_m \neq 0$ but $a_0 = a_1 = ... = a_{m-1} = 0$, then z_0 is called a zero of f of *order* or *multiplicity* m.

THEOREM 71. Suppose g is continuous in $D(z_0, R)$ and $g(z_0) = 1$. Then there is an $r \in (0, R)$ such that $|g(z)| \geq 1/2$ for all $z \in D(z_0, r)$.

THEOREM 72. If the holomorphic function f has a zero of finite order at z_0, then there is an $r > 0$ such that z_0 is the only zero of f in $D(z_0, r)$.

THEOREM 73. Suppose f is a holomorphic function on the open Ω and that Ω' is an open connected set which intersects Ω. Then there is at most one holomorphic function g on $\Omega \cup \Omega'$ which coincides with f on Ω.

HINT: Suppose g_1 and g_2 are two such functions. Then $g_1 - g_2 = 0$ on $\Omega \cap \Omega'$. By way of contradiction assume that there is a point z_1 such that $(g_1 - g_2)(z_1) \neq 0$. For a connected contour Γ (or, for simplicity, a smooth path) joining a point z_0 in $\Omega \cap \Omega'$ and z_1 consider the values of the continuous function $g_1 - g_2$ on Γ^*.

DEFINITION 35. The function $g : \Omega \cup \Omega' \to \mathbb{C}$ from Theorem 73 (if it exists) is called the *analytic continuation* of f from Ω to $\Omega \cup \Omega'$.

5.3. The global version of Cauchy's theorem

THEOREM 74 (♯ Cauchy's integral formula, global version). Suppose Ω is an open subset of the complex plane, f is a holomorphic function on Ω, and Γ is a closed contour in Ω such that $\mathrm{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω. Then

$$f(z)\,\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(u)}{u-z}du$$

for every $z \in \Omega \setminus \Gamma^*$.

THEOREM 75 (Cauchy's theorem, global version). Suppose Ω, f, and Γ satisfy the same hypotheses as in Theorem 74. Then $\int_\Gamma f = 0$.

DEFINITION 36. A connected open set Ω in the complex plane is called *simply connected* if $\mathbb{C} \setminus \Omega$ has no bounded component.

EXERCISE 31. Show that the set $\mathbb{C} \setminus \{0\}$ is not simply connected but that $\mathbb{C} \setminus (-\infty, 0]$ is.

THEOREM 76. If Ω is a non-empty open simply connected subset of $\mathbb{C}$ and Γ a closed contour in Ω, then $\mathrm{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω.

THEOREM 77 (Cauchy's theorem and integral formula for simply connected open sets). If f is a holomorphic function on the simply connected open set Ω, then

$$f(z)\,\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(u)}{u-z}du$$

for every $z \in \Omega \setminus \Gamma^*$ and

$$\int_\Gamma f = 0.$$

THEOREM 78. Suppose $0 \le r_1 < r_2$. Let $\gamma_1(t) = z_0 + r_1 \exp(\mathrm{i}t)$ and $\gamma_2(t) = z_0 + r_2 \exp(\mathrm{i}t)$, $t \in [0, 2\pi]$. If $\Gamma = \gamma_1 \ominus \gamma_2$, then $\mathrm{Ind}_\Gamma(w) = 0$ for all w satisfying either $|w - z_0| < r_1$ or $|w - z_0| > r_2$.

THEOREM 79. Suppose $0 \le r_1 < r_2$. Let $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $\gamma(t) = z_0 + r\exp(\mathrm{i}t)$, $t \in [0, 2\pi]$. If $g : A \to \mathbb{C}$ is holomorphic, show that the value of $\int_\gamma g$ does not depend on r as long as $r_1 < r < r_2$.

HINT: Use Cauchy's global theorem 75.

NOTATION 5. A series of the form $\sum_{n=-\infty}^{\infty} a_n$ is called convergent when both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are convergent. In this case

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}.$$

THEOREM 80 (Laurent series). Suppose $0 \le r_1 < r_2$ and f is holomorphic in the annulus $\Omega = \{z \in \mathbb{C} : r_1 < |z - a| < r_2\}$. Then, f can be expressed by a *Laurent series*, i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

The coefficients a_n are given by the integrals

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(u)}{(u-a)^{n+1}}du$$

where $\gamma = a + r' \exp(it)$, $t \in [0, 2\pi]$ and $r' \in (r_1, r_2)$.

The Taylor series of an analytic function is, of course, a special case of a Laurent series.

HINT: Fix $z \in \Omega$. For $j = 1, 2$ let $\gamma_j(t) = a + r_j' \exp(it)$ for $t \in [0, 2\pi]$ such that $r_1 < r_1' < |z - z_0| < r_2' < r_2$. Then consider $\int_{\gamma_2 \ominus \gamma_1} f$. To find a_n for $n \ge 0$ employ Theorem 51; for the others use a variant of its proof after noting that $u - z = -(z-a)(1-(u-a)/(z-a))$. The values for a_n are independent of r_1' and r_2' by Theorem 79.

EXERCISE 32. Let $f(z) = (z + 3)/(z - \mathrm{i})$ and $\gamma(t) = \mathrm{i} + \exp(\mathrm{i}t)$, $t \in [0, 2\pi]$. Find $\int_\gamma f$.

CHAPTER 6

Isolated singularities and the calculus of residues

Throughout this chapter Ω denotes an open subset of the complex plane.

6.1. Classifying isolated singularities

DEFINITION 37. Suppose $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then z_0 is called an *isolated singularity* of f.

EXERCISE 33. Identify sets Ω and the point z_0 to show that the following functions have an isolated singularity: (i) $z \mapsto (z^2+1)/(z-\mathrm{i})$, (ii) $z \mapsto 1/(z-3)$, and (iii) $z \mapsto \exp(1/z)$.

A punctured disk, i.e., a set of the form $\{z \in \mathbb{C} : 0 < |z - z_0| < r\}$, is a special case of an annulus and thus a holomorphic function defined on it has a Laurent expansion. Therefore, if z_0 is an isolated singularity of f, then there is a punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ (possibly the punctured plane) on which

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

DEFINITION 38. Let z_0 be an isolated singularity of the holomorphic function $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ and $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ the corresponding Laurent series. We call the point z_0 a *removable singularity*, if $a_n = 0$ for all $n < 0$. We call z_0 a *pole* of *order* m (where $m > 0$) or of *multiplicity* m, if $a_{-m} \neq 0$ but $a_n = 0$ for all $n < -m$. In any other case z_0 is called an *essential singularity*.

EXERCISE 34. Classify the isolated singularities from Exercise 33.

THEOREM 81. The point z_0 is a removable singularity of f if and only if f has a limit at z_0.

HINT: Use Theorem 62.

EXERCISE 35. Show that the function $z \mapsto f(z) = 1/(\exp(z) + \exp(-z))^2$ has a pole at $\mathrm{i}\pi/2$. Determine the order m of the pole as well as a number a and a function g such that g is holomorphic in $D(\mathrm{i}\pi/2, \pi)$, $g(0) = 0$, and

$$f(z) = a(z - z_0)^{-m}(1 + g(z)).$$

THEOREM 82. Suppose f has the Laurent expansion $\sum_{n=m}^{\infty} a_n(z - z_0)^n$ in the disk $D(z_0, R)$ for some $R > 0$ where $m \in \mathbb{Z}$ and $a_m \neq 0$. Then

$$f(z) = a_m(z - z_0)^m(1 + g(z))$$

for some function g which is holomorphic in $D(z_0, R)$ and vanishes at z_0.

THEOREM 83. The point z_0 is a pole of order m of a function f if and only if $1/f$ has an analytic continuation for which z_0 is a zero of order m.

HINT: Recall that, if g is holomorphic near z_0 and $g(z_0) = 0$, then $z \mapsto 1/(1 + g(z))$ is holomorphic near z_0.

THEOREM 84. If z_0 is a pole of f and M is any positive real number, then there is a positive δ such that $|f(z)| \geq M$ for all $z \in D(z_0, \delta) \setminus \{z_0\}$.

HINT: Note that, if g is continuous and $g(z_0) = 0$ we have $|1 + g(z)| \geq 1/2$ in any sufficiently small disk about z_0.

THEOREM 85. If z_0 is an isolated singularity of f and there is a natural number m such that $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$, then z_0 is either a removable singularity of f or else a pole of order at most m.

DEFINITION 39. Let P be a set of isolated points in Ω without a limit point in Ω and f a holomorphic function on $\Omega \setminus P$. If no point of P is an essential singularity of f, then f is called *meromorphic* on Ω.

THEOREM 86 ($\sharp$ The Great Picard theorem). Suppose f is a holomorphic function defined on the punctured disk $D' = D(z_0, r) \setminus \{z_0\}$. If z_0 is an essential singularity of f, then f assumes all complex values, with at most one exception, infinitely often.

EXERCISE 36. Show that 0 is an essential singularity of the function $z \mapsto \exp(1/z)$ and that the Great Picard theorem holds.

6.2. The calculus of residues

DEFINITION 40. If the holomorphic function f has the isolated singularity z_0 and the Laurent expansion $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, then the number

$$a_{-1} = \frac{1}{2\pi i} \int_\gamma f,$$

where $\gamma(t) = z_0 + r\exp(it)$, $t \in [0, 2\pi]$ for sufficiently small r, is called the *residue* of f at z_0 and is denoted by $\operatorname{Res}(f, z_0)$.

THEOREM 87. If $\lim_{z \to z_0} (z - z_0) f(z) = a \neq 0$, then z_0 is a *simple pole*, i.e., a pole of order one, of f and $a = \operatorname{Res}(f, z_0)$.

THEOREM 88 (Cauchy's residue theorem). Suppose Ω is an open subset of the complex plane, f is a holomorphic function on $\Omega' = \Omega \setminus \{z_1, ..., z_n\}$, and Γ is a closed contour in Ω' such that $\operatorname{Ind}_\Gamma(w) = 0$ whenever w is not an element of Ω (i.e., Γ does not wind around any point outside Ω). Then

$$\int_\Gamma f = 2\pi \mathrm{i} \sum_{k=1}^{n} \operatorname{Res}(f, z_k) \operatorname{Ind}_\Gamma(z_k).$$

HINT: Let $m_k = \operatorname{Ind}_\Gamma(z_k)$ and $\Gamma' = \bigoplus_{k=1}^n \gamma_k$ where $\gamma_k(t) = z_k + r_k \exp(\mathrm{i} m_k t)$ for $t \in [0, 2\pi]$ for sufficiently small but positive r_k. Recalling Theorem 57 we obtain $\int_\Gamma f = \int_{\Gamma'} f$.

EXERCISE 37. Compute $\int_{-\infty}^{\infty} 1/(1 + x^2) dx$ with the tools of Calculus as well as the residue theorem.

HINT: Consider a contour made up from the interval $[-R, R]$ and a semi-circle of radius R.

EXERCISE 38. Compute $\int_{-\infty}^{\infty} 1/(\exp(x) + \exp(-x)) dx$.

HINT: Consider a rectangle with vertices $\pm R$ and $\pm R + \pi i$.

Exercise 39. Compute $\int_0^{2\pi} 1/(2+\sin x)dx$.

Hint: Let $z = \exp(it)$ and note that $2i\sin(t) = z - 1/z$.

Exercise 40. If $m \in \mathbb{Z}$ let $f(z) = (z - z_0)^m$. Determine the poles of f'/f. Find their order and their residues.

Theorem 89. Suppose f is a meromorphic function on Ω. Show that f'/f is also meromorphic on Ω, that its poles are all simple, and that they occur precisely at the poles and zeros of f. Moreover $\operatorname{Res}(f'/f, z) = m$ if z is a zero of order m or a pole of order $-m$.

Notation 6. Suppose f is a meromorphic function on Ω. We denote the sets of its zeros and poles by Z_f and P_f, respectively. For $z_0 \in \Omega$ define $M_f(z_0)$ to be the smallest integer m such that coefficient a_m in the Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is non-zero. The number of points in a set S is denoted by $\#S$, if it is finite.

Note that z_0 is a pole of f of order $-M_f(z_0)$ if and only if $M_f(z_0) < 0$. z_0 is a zero of f of order $M_f(z_0)$ if and only if $M_f(z_0) > 0$.

Theorem 90 (Counting zeros and poles). Suppose f is a meromorphic function with finitely many zeros and poles. Let Γ be a closed contour in $\Omega\setminus(Z\cup P)$ such that $\operatorname{Ind}_\Gamma(w) = 0$ whenever $w \in \mathbb{C}\setminus\Omega$. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f} = \sum_{z\in Z\cup P} M_f(z)\operatorname{Ind}_\Gamma(z).$$

In particular, if f is holomorphic and $\Gamma(t) = z_0 + r\exp(\mathrm{i}t)$, $t \in [0, 2\pi]$, then $\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}$ is the number of zeros (counted according to their multiplicities) in $D(z_0, r)$.

Theorem 91. Suppose a and b are complex numbers satisfying $|a-b| < |a|+|b|$. Then neither a nor b can be 0 and a/b cannot be negative.

Exercise 41. Suppose γ and h are defined by $\gamma(t) = \exp(\mathrm{i}t)$, $t \in [0, 2\pi]$, and $h(z) = \mathrm{i}z^2 + z + 4$. Show that $h\circ\gamma$ is a closed smooth path in the half plane $\{z : \operatorname{Re}(z) > 0\}$. Find $\operatorname{Ind}_{h\circ\gamma}(0)$.

Theorem 92. Suppose γ is a closed smooth path in Ω and h is meromorphic on Ω with no poles on γ^*. Then $h\circ\gamma$ is a smooth path in $\mathbb{C}$. Moreover, if h does not take any value in $(-\infty, 0]$, then $\int_\gamma h'/h = \operatorname{Ind}_{h\circ\gamma}(0) = 0$.

Hint: Use Theorem 56.

Theorem 93 (Rouché's theorem). Let f and g be meromorphic functions on Ω and assume that $\overline{D(a,r)} \subset \Omega$. If no zero or pole lies on the circle $C = \{z : |z-a| = r\}$ and if $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in C$, then $\#Z_f - \#P_f = \#Z_g - \#P_g$.

Hint: Let $\gamma : [0, 2\pi] \to \Omega : t \mapsto a + r\exp(\mathrm{i}t)$, set $h = f/g$, and employ Theorems 91 and 92.

Exercise 42. Determine the number of zeros of $z \mapsto f(z) = z^6 - 3z^5 + z^2 - 9z + 3$ in the disks $D(0,1)$ and $D(0,2)$. The key here is to choose g as one of the monomials occurring in f and use the triangle inequality.

CHAPTER 7

A zoo of functions

In this chapter we investigate briefly the most elementary functions of analysis. The exponential functions has been introduced earlier since it is too important to postpone its use.

7.1. Polynomial and rational functions

Definition 41. If n is a non-negative integer and a_0, a_1, …, a_n are complex numbers, then the function $p : \mathbb{C} \to \mathbb{C}$ defined by

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

is called a *polynomial function* (or polynomial for short). The integer n is called the *degree* of p if a_n is different from 0. The number a_n is then called the *leading coefficient* of p. The zero function is also a polynomial but no degree is assigned to it.

Theorem 94. Any polynomial is an entire function.

Theorem 95. Let p be a polynomial of degree n and leading coefficient a_n. If $k \in \mathbb{N}$ and $k \leq n$, then $p^{(k)}$ is a polynomial of degree $n - k$. In particular, $p^{(n)}$ is identically equal to $n!a_n$. If $k > n$, then $p^{(k)}$ is identically equal to 0.

Theorem 96. Let p be a polynomial of degree n and leading coefficient a_n. The Taylor series of the polynomial p about any point $z_0 \in \mathbb{C}$ is a finite sum, in fact,

$$p(z) = \sum_{k=0}^{n} b_k (z - z_0)^k$$

for appropriate coefficients b_k. In particular, $b_n = a_n$.

Theorem 97. Let p be a polynomial of degree $n \geq 1$ and z_0 a zero of p. Then there is a polynomial q of degree $n - 1$ such that $p(z) = (z - z_0)q(z)$ for all $z \in \mathbb{C}$.

Theorem 98 (The fundamental theorem of algebra). Suppose p is a polynomial of degree $n \geq 1$. Then there exist numbers a and z_1, ..., z_n (not necessarily distinct) such that

$$p(z) = a \prod_{k=1}^{n} (z - z_k).$$

Hint: One proof uses the fact that $1/p$ would be entire if p had no zeros and study its behavior near infinity. Another uses Rouché's theorem.

Theorem 99. The coefficients a_k of a polynomial with leading coefficient 1 vary continuously with the zeros.

That the converse is also true can be proved with the aid of Rouché's theorem. The precise statement is as follows:

THEOREM 100 (♯). Suppose $f(z) = \sum_{k=0}^{n} a_k z^k$ and $g(z) = \sum_{k=0}^{n} b_k z^k$ are two polynomials with $a_n = b_n = 1$. If z_0 is the only zero of f in $\overline{D(z_0, r)}$ and if the multiplicity of z_0 is m then the following holds: For every $\varepsilon \in (0, r)$ there is a $\delta > 0$ such that, if $|a_k - b_k| < \delta$ for $k = 0, ..., n-1$, then g has precisely m zeros in $D(z_0, \varepsilon)$ (counting multiplicities).

DEFINITION 42. Let p and q be polynomials and $Q = \{z \in \mathbb{C} : q(z) = 0\}$. Assume $Q \neq \mathbb{C}$, i.e., q is not the zero polynomial. Then, the function $r : (\mathbb{C} \setminus Q) \to \mathbb{C}$ given by

$$r(z) = \frac{p(z)}{q(z)}$$

is called a *rational function*.

THEOREM 101 (Rational functions). A rational function is a meromorphic function on $\mathbb{C}$. Its poles are zeros of q. Conversely, a zero z_0 of q is a pole if and only if $M_p(z) < M_q(z)$.

DEFINITION 43. A rational function of the type $z \mapsto (az+b)/(cz+d)$ where $ad - bc \neq 0$ is called a *Möbius transform*.

NOTATION 7. The set $\mathbb{C} \cup \{\infty\}$, called the extended complex plane, is denoted by $\mathbb{C}_\infty$. One may extend a Möbius transform as function from $\mathbb{C}_\infty$ to itself in the following way: If $c \neq 0$ set

$$M(z) = \begin{cases} (az+b)/(cz+d) & \text{if } z \in \mathbb{C} \setminus \{-d/c\} \\ \infty & \text{if } z = -d/c \\ a/c & \text{if } z = \infty. \end{cases}$$

If $c = 0$ we must have $d \neq 0$ and set

$$M(z) = \begin{cases} (az+b)/d & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

THEOREM 102 (♯). A Möbius transform may be interpreted as a bijective function from $\mathbb{C}_\infty$ to itself. Conversely, an injective meromorphic function on $\mathbb{C}_\infty$ is a Möbius transform. The set of all Möbius transforms forms a group under composition.

Möbius transforms have many interesting properties to which an entire chapter might be devoted.

7.2. Trigonometric and hyperbolic functions

DEFINITION 44. The *sine*, *cosine*, and *tangent* function, denoted by sin, cos, and tan, respectively, are defined by

$$\sin(z) = \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}.$$

$$\cos(z) = \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2},$$

and

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \mathrm{i}\frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}.$$

THEOREM 103. cos and sin have the following properties:

(1) They are entire functions.

(2) Their derivatives are $\sin'(z) = \cos(z)$ and $\cos'(z) = -\sin(z)$.
(3) The Pythagorean theorem holds: $(\sin z)^2 + (\cos z)^2 = 1$ for all $z \in \mathbb{C}$.
(4) Addition theorems:

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

and

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

for all $z, w \in \mathbb{C}$.
(5) Taylor series:

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

and

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

In particular, sin and cos are extensions of the functions defined in Real Analysis.
(6) $\cos(z) = \sin(z + \pi/2)$.
(7) $\sin(z) = 0$ if and only z is an integer multiple of π.
(8) The range of both, the sine and the cosine function, is $\mathbb{C}$.

Hint: For (8) let w be an arbitrary element of $\mathbb{C}$ and consider $\sin(z) = w$. Then $u = \exp(\mathrm{i}z)$ satisfies a quadratic equation. Why do we know it has a (non-zero) solution even if we don't know how to find it? Finally use Theorem 20.

Theorem 104. The tangent function is a meromorphic function on $\mathbb{C}$. All its poles are simple and occur precisely at the points $(n + 1/2)\pi$, $n \in \mathbb{Z}$.

Definition 45. The *hyperbolic sine* and *hyperbolic cosine* function, denoted by sinh and cosh, respectively, are defined by

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2} \text{ and } \cosh(z) = \frac{\exp(z) + \exp(-z)}{2}.$$

The *hyperbolic tangent* function, denoted by tanh, is defined by $\tanh(z) = \sinh(z)/\cosh(z)$.

Theorem 105. The following relationships hold: $\sinh(z) = -\mathrm{i}\sin(\mathrm{i}z)$, $\cosh(z) = \cos(\mathrm{i}z)$, $\tanh(z) = -\mathrm{i}\tan(\mathrm{i}z)$.

7.3. The logarithmic function

Theorem 106 (✓). Since $\exp : \mathbb{R} \to (0, \infty)$ is a bijective function, it has an inverse $\ln : (0, \infty) \to \mathbb{R}$. We have that $\ln(1) = 0$ and $\ln'(x) = 1/x$ so that, by the fundamental theorem of calculus

$$\ln(x) = \int_1^x \frac{dt}{t}$$

for all $x > 0$.

Definition 46. Suppose Ω is a simply connected non-empty open subset of $\mathbb{C} \setminus \{0\}$ and Γ is a contour in $\mathbb{C} \setminus \{0\}$ connecting 1 to $z_0 \in \Omega$. Then we define the function $L_\Gamma : \Omega \to \mathbb{C}$ by setting

$$L_\Gamma(z) = \int_{\Gamma \oplus \beta} \frac{du}{u}$$

where β is a contour in Ω connecting z_0 to z. As the notation suggests $L_\Gamma(z)$ does not depend on β as long as β remains in Ω. The function L_Γ is called a *branch of the logarithm* on Ω.

EXERCISE 43. Let $r > 0$ and $\theta \in (-\pi, \pi]$ be the polar coordinates of the non-zero complex number $z = r\exp(\mathrm{i}\theta)$. Let $\gamma(t) = (1-t) + rt$ and $\eta_\phi(t) = r\exp(\mathrm{i}\phi t)$ for $t \in [0,1]$. Compute $\int_{\gamma\oplus\eta_\theta} du/u$ and $\int_{\gamma\oplus\eta_{\theta+4\pi}} du/u$.

THEOREM 107. Let Ω be a non-empty, open, simply connected subset of $\mathbb{C}$. Suppose that Γ and Γ' are two contours in $\mathbb{C}\setminus\{0\}$ connecting 1 to points z_0 and z_0' in Ω, respectively. If γ is a contour in Ω connecting z_0 and z_0', then

$$L_\Gamma(z) - L_{\Gamma'}(z) = 2\pi\mathrm{i}\,\mathrm{Ind}_{\Gamma\oplus\gamma\ominus\Gamma'}(0).$$

Thus there are (at most) countably many different functions L_Γ which are defined this way, even though there are many more contours Γ connecting 1 to a point in Ω.

THEOREM 108. Let Ω and Γ be as in Theorem 107. For each $m \in \mathbb{Z}$ there is a Γ' connecting 1 to z_0 such that $L_\Gamma(z) - L_{\Gamma'}(z) = 2m\pi\mathrm{i}$.

This shows that defining the logarithm as an antiderivative comes with certain difficulties similar to those of defining the inverse sine function in Calculus. At the same time this behavior gives rise to a lot of interesting mathematics.

THEOREM 109. L_Γ is a holomorphic function on Ω with derivative $1/z$.

THEOREM 110. $\exp(L_\Gamma(z)) = z$ for all $z \in \Omega$ but $L_\Gamma(\exp(z))$ may well differ from z by an integer multiple of $2\pi i$.

HINT: The function $z \mapsto z\exp(-L_\Gamma(z))$ has zero derivative. To compute $L_\Gamma(z_0)$ use Exercise 43 and Theorem 107.

DEFINITION 47. Let $\Omega = \mathbb{C}\setminus(-\infty, 0]$ and $\Gamma : [0,1] \to \mathbb{C} : t \mapsto 1)$. We call the function $\log = L_\Gamma$ the *principal branch* of the *logarithm*.

THEOREM 111. If $z \in \mathbb{C}\setminus(-\infty,0]$ has polar representation $z = r\exp(it)$ where $t \in (-\pi,\pi)$ and $r > 0$, then

$$\log(z) = \ln(r) + it.$$

The range of log is the strip $\{z \in \mathbb{C} : |\,\mathrm{Im}(z)| < \pi\}$. In particular, $\log(\exp(z)) = z$ if and only if $|\,\mathrm{Im}(z)| < \pi$.

THEOREM 112. The Taylor series of $z \mapsto \log(1+z)$ about $z = 0$ has radius of convergence 1 and is given by

$$\log(1+z) = -\sum_{n=1}^{\infty} \frac{(-z)^n}{n}.$$

THEOREM 113. If $\mathrm{Re}(a)$ and $\mathrm{Re}(b)$ are positive, then $\log(ab) = \log(a) + \log(b)$.

EXERCISE 44. Find two numbers a and b such that $\log(ab) \neq \log(a) + \log(b)$.

7.4. Power functions

THEOREM 114. If L is any branch of the logarithm, a is in the domain of L, and $b \in \mathbb{Z}$, then $a^b = \exp(bL(a))$.

DEFINITION 48. Suppose Ω is a simply connected non-empty open subset of $\mathbb{C} \setminus \{0\}$ and $L : \Omega \to \mathbb{C}$ a branch of the logarithm. For a fixed complex number p the function $\Omega \to \mathbb{C} : z \mapsto z^p = \exp(pL(z))$ is called a *branch of the power p function*.

In general, the value of z^p depends on the branch of the logarithm chosen and is therefore ambiguous when $p \notin \mathbb{Z}$. In most cases one chooses, of course, the principal branch of the logarithm and defines $a^b = \exp(b \log(a))$ (forsaking the definition of powers of negative numbers). In particular, defining $\mathrm{e} = \exp(1)$, we get $\mathrm{e}^z = \exp(z)$ for all $z \in \mathbb{C}$.

EXERCISE 45. Compute all possible values of i^i.

EXERCISE 46. Choose a branch of the logarithm which includes the negative real axis and compute $(-2)^{1/2}$. What other values can one obtain by choosing different branches?

THEOREM 115. Each branch of the power p function is a holomorphic function which never vanishes.

THEOREM 116. Suppose $a, b, p \in \mathbb{C}$. Then $z^a z^b = z^{a+b}$, in particular, $1/z^p = z^{-p}$.

EXERCISE 47. Pick a point z in the second quadrant of $\mathbb{C}$ and show that $(z^2)^{1/2} \neq z$ using the principal branch of the power 1/2-function.

THEOREM 117. The following statements hold.

(1) If $p \in \mathbb{N}_0$ all branches of the power p function may be analytically extended to the entire complex plane and any branch gives then rise to one and the same function.
(2) The same is true if $p \in \mathbb{Z}$ is negative, except that the function may not be extended to 0 since 0 is then a pole of the function.
(3) If $p \in \mathbb{Q}$ there are only finitely many different branches of $z \mapsto z^p$ in any simply connected open set not containing 0. In fact, if m/n is a representation of p in lowest terms, there will be n branches of $z \mapsto z^p$.

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