Joint Program Exam in Real Analysis

May 3, 2021

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and **justify carefully your reasoning.** Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you have used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mu(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^n$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of $d\mu$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\| \cdot \|_p$ denotes the norm on $L^p(X, \mu)$. The notation $\mu(A)$ it is also denoted as $\mathrm{m}(A)$, indicating the Lebesgue measure of the set A.

1. Let $f \in L^1(\mathbb{R}^n)$, f be Lebesgue measurable, $t \in \mathbb{R}^n$, $g(x) = f(x-t)$. Prove that:
$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} g d\mu.$$

2. Let $f : [a,b] \to \mathbb{R}$ whose derivative f' exists a.e in $[a,b]$. Suppose there exists $c > 0$ such that $|f'(t)| \leq c$ a.e in $[a,b]$. Prove or disprove
$$f(x) - f(a) = \int_a^x f'(t) dt.$$

3. Let $1 < p < \infty$, $f \in L^p(0,\infty)$, $p^{-1} + q^{-1} = 1$, $F(x) = \int_{(0,x)} f \; d\mu$. Show that:
$$\lim_{x \to 0} x^{-1/q} F(x) = 0.$$

4. Show that there is no set $E \subset [0,1]$ such that
$$\mu([x,1] \cap E) = (1-x)^2 \quad \text{for all } x \in [0,1].$$

5. Let $\{f_k\}$ be a sequence of measurable functions which converges a.e. in $[0,1]$ to a finite limit f. Prove that there exists a subset E of $[0,1]$ with Lebesgue measure $\mu(E) > 3/4$ such that
$$\int_E f = \lim_{k \to \infty} \int_E f_k.$$

6. Explain whether or not this limit may be evaluated. If you may, compute its value with the proper justification.
$$\lim_{n \to \infty} \int_0^n x[1 - \frac{x}{n}]^{\ln(n)} \; dx.$$

7. Let $\{f_k\}$ be a sequence in L^p with $1 \leq p < \infty$. If $\lim_{k \to \infty} \|f_k - f\|_p = 0$, show that $f_k \to f$ in measure.

8. Let $f \in L^1(0,1)$ and $g(x) = \int_x^1 t^{-1} f(t) \, dt$ for $x \in (0,1]$. Show :
   (a) g is measurable;
   (b) $\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx$.