Joint Program Exam in Real Analysis

September 2020

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All the problems are weighted equally. You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout the exam all the integrals mean the Lebesgue integrals. $L^p(E)$ denotes the L^p space with respect to Lebesgue measure on the Lebesgue measurable set E and $\|\cdot\|_p$ denotes the norm on $L^p(E)$. $[a, b]$ denotes a bounded and closed interval in $\mathbb{R}$.

1. Determine whether each of the following statements is ture or false? Justify your answers.

   (a) If $f \in L^1([0,\infty))$ and $f \geq 0$ then $\lim_{x\to\infty} f(x) = 0$.

   (b) If $\lim_{x\to\infty} f(x) = 0$ and $f \geq 0$ on $[0,\infty)$ then $f \in L^1([0,\infty))$.

   (c) If f is a measurable function on $[a,b]$ with $f(x) > 0$ for almost every $x \in [a,b]$, then $\int_a^b f(x)\,dx > 0$.

2. Assume $f_n : E \to \overline{\mathbb{R}}$ ($\overline{\mathbb{R}}$ is the set of extended real numbers) is a sequence of measurable functions with E measurable. Prove that

$$A = \{x \in E : \{f_n(x)\}_{n=1}^{\infty} \text{ does not converge}\}$$

   is measurable.

3. Determine $\lim_{n\to\infty} \int_0^n (1 + x/n)^n \mathrm{e}^{-2x} dx$ and justify your answer.

4. Let $f, g \in L^3(E)$ for $E \subseteq \mathbb{R}^n$ and

$$\|f\|_3 = \|g\|_3 = \int_E f^2(x) g(x)\,dx = 1.$$

   Prove that $|f(x)| = g(x)$ a.e. in E.

5. Let f be nonnegative and $f \in L^1([0,1])$. Let $0 < \alpha < 1$. Define

$$g(x) := \int_0^1 \frac{f(y)}{|x-y|^\alpha}\,dy \qquad \forall\, x \in [0,1].$$

   Prove $g \in L^1([0,1])$ and estimate $\|g\|_1$ in terms of $\|f\|_1$.

6. Let

$$f(x) = \begin{cases} x^\alpha \cos \dfrac{1}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

   Prove that f is of bounded variation on $[0,1]$ if and only if $\alpha > 1$.

7. Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of absolutely continuous functions defined on $[0,1]$ and $f_n(0) = 0$ $(n = 1, 2, \cdots)$. Assume also that $\{f_n'\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^1([0,1])$. Prove that there exists some absolutely continuous function f on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$.

8. Prove the following statements:

   (a) Any monotone function f on $\mathbb{R}$ has at most countably many points of discontinuity.

   (b) If f is a monotone function on $[0,1]$ and $f(1) = f(0) + \int_0^1 f'(x)\,dx$, then f is absolutely continuous on $[0,1]$.