Joint Program Exam in Real Analysis

May 2017

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All the problems are weighted equally. **You need to do 7 of the 8 problems for full credit.**

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout this exam m and m_n denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$, respectively, or on one of their subsets. The words 'measurable' and 'integrable' refer to Lebesgue measure and all integrals are Lebesgue integrals. Integration with respect to m is also denoted by dx (or similar). $L^p(E)$ denotes the L^p-space with respect to Lebesgue measure on the Lebesgue measurable set E, with corresponding norm $\|\cdot\|_p$.

1. For each of the following statements decide whether it is true or false. If true, give a proof; if false, give a counter-example.

   (a) $L^1(\mathbb{R}) \cap L^3(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

   (b) $L^2(\mathbb{R}) \subseteq L^1(\mathbb{R}) \cup L^3(\mathbb{R})$.

2. Find

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)}\, dx.$$

   Justify your answer.

3. Let $1 \le p < \infty$, $M < \infty$ and $f_k \to f$ in $L^p(\mathbb{R})$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \le M$ for all k. Show that $f_k g_k \to fg$ in $L^p(\mathbb{R})$.

4. Let $f : [0,1] \to \mathbb{C}$ be absolutely continuous and $f' \in L^p(0,1)$ for some $p \in (1,\infty)$. Show that f is Hölder-continuous with exponent $\frac{p-1}{p}$, i.e., there exists a finite constant C such that

$$|f(x) - f(y)| \le C|x-y|^{(p-1)/p} \quad \text{for all } x, y \in [0,1].$$

   What can be said in the case $p = \infty$?

5. Suppose that $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, where $g(x) = x^2 f(x)$. Let $h(x) = xf(x)$. Prove that (a) $h \in L^2(\mathbb{R})$ and (b) $h \in L^1(\mathbb{R})$.

6. Let $A, B \subset \mathbb{R}$ be measurable and $0 < m(A) < \infty$, $0 < m(B) < \infty$. Prove that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_B(y) \chi_{A+x}(y)\, dy\, dx = m(A)m(B),$$

   and that there exists an x_0 such that $m(B \cap (A + x_0)) > 0$.

   (Hint: Rewrite $\chi_{A+x}(y)$ as a characteristic function in the variable x.)

7. Let $\phi \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx \right)^{1/n} = \|\phi\|_\infty.$$

8. Let f be a measurable function defined on a measurable $E \subseteq \mathbb{R}^n$.

   (a) For $p \in (0, \infty)$, prove that

$$\int_E |f|^p\, dm_n = p \int_0^\infty t^{p-1}\, m_n(\{x \in E : |f(x)| > t\})\, dt.$$

(b) Suppose that $|f(x)| \leq M$ almost everywhere, and

$$m_n(\{x \in E : |f(x)| > t\}) \leq Kt^{-2}$$

for all $t > 0$. Prove that $f \in L^p(E)$ for all $p \in (2, \infty)$.