Joint Program Exam in Real Analysis

May 2016

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout this exam all the integrals are Lebesgue integrals. $L^p(E)$ denotes the L^p-space with respect to Lebesgue measure on the Lebesgue measurable set E, with corresponding norm $\|\cdot\|_p$. For general measure spaces we abbreviate $L^p(X, \mu)$ by L^p. By m_n we denote Lebesgue measure on $\mathbb{R}^n$.

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1. Suppose $E \subset \mathbb{R}$ is a set of Lebesgue measure zero. Prove that there exists $x \in \mathbb{R}$ such that $x + E \subset \mathbb{R} \setminus \mathbb{Q}$.

2. (a) Prove that, for any measure space, if $f \in L^1 \cap L^\infty$ then $f \in L^2$ and $\|f\|_2^2 \leq \|f\|_1 \|f\|_\infty$.

   (b) Suppose $\{a_1, \ldots, a_N\}$ is a finite sequence satisfying $a_k = \pm 1$ for all $1 \leq k \leq N$. Prove that

$$\int_0^1 \Big| \sum_{k=1}^N a_k e^{2\pi i k\theta} \Big| d\theta \geq 1.$$

3. If $f \in L^1([0, \infty))$, prove that

$$\lim_{R \to \infty} \frac{1}{R} \int_{[0,R]} x f(x) dx = 0.$$

4. (a) Let f be measurable on $\mathbb{R}^n$ with respect to Lebesgue-measure m_n and $0 < p < \infty$. Show that

$$\int_{\mathbb{R}^n} |f|^p \, dm_n = p \int_0^\infty t^{p-1} \, m_n \left(\{x : |f(x)| > t\} \right) \, dt.$$

   (b) Use the result from (a) to calculate $\displaystyle\int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dm_2$.

5. Let f_n be absolutely continuous on $[a, b]$, $f_n(a) = 0$. Suppose f_n' is a Cauchy sequence in $L^1([a, b])$. Show that there exists f absolutely continuous on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and periodic with period 1: $f(t + 1) = f(t)$, $t \in \mathbb{R}$. Suppose that there is a finite c such that $\int_{[0,1]} |f(a + t) - f(b + t)| dt \leq c$ for all a and b. Show that $f \in L^1[0, 1]$.

7. If $f_n \to f$ in measure and $|f_n| < g \in L^p$, $1 \leq p < \infty$, then $\|f_n - f\|_p \to 0$. Prove.