Joint Program Exam in Real Analysis

May 5, 2015

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally. You need to do all of the problems for full credit.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure. Instead of $d\mathrm{m}$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\|\cdot\|_p$ denotes the norm on $L^p(X, \mu)$.

1. Suppose that f is a bounded measurable function and that the improper Riemann integral $\int_0^\infty f(x)\,dx$ exists (as a finite number R).

   (a) Show that f is Lebesgue integrable and that $R = \int_{(0,\infty)} f\,d\mathrm{m}$, if $f \geq 0$.

   (b) Show that the conclusion in (a) may be false without the hypothesis $f \geq 0$.

   (c) Find a bounded Lebesgue integrable function on $[0,1]$ for which the Riemann integral does not exist.

2. Are the following statements true or false? Justify your answers.

   (a) If $f \in L^1(X,\mu)$, then $Y = \{x \in X : f(x) \neq 0\}$ is σ-finite.

   (b) A sequence of measurable functions on $[0,1]$ which converges to 0 pointwise almost everywhere converges to 0 in $L^1([0,1],\mathrm{m})$.

   (c) If $f \in L^p(\mathbb{R}^n,\mathrm{m}) \cap L^\infty(\mathbb{R}^n,\mathrm{m})$ and $q > p$, then $f \in L^q(\mathbb{R}^n,\mathrm{m})$.

3. Suppose that f is continuous and bounded on $\mathbb{R}$. Compute

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \frac{1}{1+n^2x^2}\,dx$$

   and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \frac{n}{1+n^2x^2}\,dx.$$

4. Prove that $\int_0^\infty \sqrt{2+\sin(x)}\mathrm{e}^{-x}\,dx < \sqrt{\frac{5}{2}}$. Hint: the antiderivative of $2\sin(x)\mathrm{e}^{-x}$ is $-(\cos(x)+\sin(x))\mathrm{e}^{-x}$.

5. Let $E = [1,\infty)$ and suppose $f \in L^2(E,\mathrm{m})$ with $f \geq 0$. Define $g(x) := \int_E f(y)e^{-xy}\,dy$ for all $x \in E$. Show that $g \in L^1(E,\mathrm{m})$ and

$$\|g\|_1 \leq \frac{1}{\mathrm{e}\sqrt{2}}\|f\|_2.$$

6. Suppose that f is a real-valued function of bounded variation on $[0,1]$, i.e.,

$$\sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : 0 = x_0 < x_1 < ... < x_n = 1\} < \infty.$$

Prove that f has limits from the right at every point in $[0, 1)$, i.e., show that $\lim_{x\downarrow c} f(x)$ exists when $0 \le c < 1$, justifying any fact needed about bounded variation from its definition.

7. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and let $E \subset [a, b]$ have Lebesgue measure zero. Show that the image set $f(E)$ also has Lebesgue measure zero.

8. Suppose μ is a positive measure on a space X and that $f_n, f \in L^1(X, \mu)$. Prove that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ if $f_n \to f \in L^1(X, \mu)$ pointwise almost everywhere and $\|f_n\|_1 \to \|f\|_1$.