Joint Program Exam in Real Analysis

September 9, 2014

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.
2. You may use up to three and a half hours to complete this exam.
3. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.
4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.
5. Throughout this exam all the integrals are Lebesgue integrals. $L^p(E)$ denotes the L^p-space with respect to Lebesgue measure on the Lebesgue measurable set E, with corresponding norm $\|\cdot\|_p$. $m(E)$ means the Lebesgue measure of the Lebesgue measurable set E and χ_E is the characteristic function of E. $[a, b]$ and (a, b) denote bounded and closed/open intervals in $\mathbb{R}$. The terms 'measurable' and 'almost everywhere' refer to Lebesgue measure.

1. Are the following statements true or false? Justify! If a statement is false, provide a counterexample.

   (a) If $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ are such that $A \cup B$ is measurable, then A and B are measurable.

   (b) If f is a non-negative function on $\mathbb{R}$ such that $\sqrt{f}$ is measurable, then f is measurable.

   (c) If f is continuous on $[0,1]$ and $f'(t) = 0$ for almost every t, then $f(1) = f(0)$.

2. Let $f_n \in L^2(0,1)$ for all $n \in \mathbb{N}$. Prove or disprove:

   (a) If $\|f_n\|_1 \to 0$, then $\|f_n\|_2 \to 0$.

   (b) If $\|f_n\|_2 \to 0$, then $\|f_n\|_1 \to 0$.

3. Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^{\pi/2} \sqrt{n \sin \frac{x}{n}}\, dx.$$

4. Let $f \in L^1(\mathbb{R})$ be non-negative, $A := \{x \in \mathbb{R} : f(x) = 1\}$ and $B := \{x \in \mathbb{R} : f(x) > 1\}$.

   (a) Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f^n \, dm = m(A) + \infty \cdot m(B).$$

   (b) If there is a constant $C < \infty$ such that $\int_{\mathbb{R}} f^n \, dm = C$ for all $n \in \mathbb{N}$, then $f = \chi_A$ almost everywhere for some measurable $A \subset \mathbb{R}$.

5. Let $f \in L^2(0,1)$ and define a function $g : (0,1) \to \mathbb{R}$ by

$$g(x) = \int_0^x \frac{f(t)}{\sqrt{1-t^2}}\, dt, \quad x \in (0,1).$$

   Prove that $g \in L^2(0,1)$ and find an explicit constant C, independent of f, such that $\|g\|_2 \le C\|f\|_2$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous if there exists a constant $C > 0$ such that $|f(x) - f(y)| \le C|x - y|$ for all $x, y \in \mathbb{R}$.

   Show that f is Lipschitz continuous (with constant C) if and only if f is absolutely continous and $|f'(x)| \le C$ for almost every $x \in \mathbb{R}$.

7. Let f be a positive measurable function defined on a measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$. Prove that

$$\left(\int_E f\,dm\right)\left(\int_E \frac{1}{f}\,dm\right) \geq m^2(E),$$

and that equality in the above inequality holds if and only if $f(x) = c$ almost everywhere in E for some constant $c > 0$.