Joint Program Exam in Real Analysis

May 13, 2014

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout the exam all the integrals mean the Lebesgue integrals. $L^p(E)$ denotes the L^p space with respect to Lebesgue measure on the Lebesgue measurable set E. $m(E)$ means the Lebesgue measure of the Lebesgue measurable set E. $[a,b]$ denotes a bounded and closed interval in $\mathbb{R}$.

1. Are the following statements true or false? Justify your answers.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

   (b) There exists a non-measurable $f \geq 0$ on $\mathbb{R}$, such that $\sqrt{f}$ is measurable.

   (c) $A \times B$ is a Lebesgue-measurable subset of $\mathbb{R}^2$ if and only if A and B are measurable subsets of $\mathbb{R}$.

2. Find
$$\lim_{n\to\infty} \int_0^\infty \frac{nx^n}{0.1 + x^{n+3}} \sin\frac{x}{n}\, dx,$$
and justify your answer.

3. Suppose that $f \in L^1(\mathbb{R})$. Prove that
$$\lim_{n\to\infty} n\, m(\{x \in \mathbb{R} : |f(x)| > n\}) = 0.$$

4. Verify that each of the integrals
$$\int_0^1 \left\{ \int_1^\infty (e^{-xy} - 2e^{-2xy})\, dx \right\} dy \quad \text{and} \quad \int_1^\infty \left\{ \int_0^1 (e^{-xy} - 2e^{-2xy})\, dy \right\} dx$$
exists, but they are unequal. What can you conclude about the function $f(x, y) = e^{-xy} - 2e^{-2xy}$ on $[1, \infty) \times [0, 1]$?

5. Let $f : [a, b] \to \mathbb{C}$ and $g : [a, b] \to \mathbb{C}$ be absolutely continuous.

   (a) Prove that fg is absolutely continuous on $[a, b]$.

   (b) Prove the integration-by-parts formula
$$\int_a^b f'g\, dx = f(b)g(b) - f(a)g(a) - \int_a^b fg'\, dx.$$

6. Suppose that $f,\ f_1,\ f_2,\ \dots \in L^2(\mathbb{R})$, $\|f_n\|_2 \to \|f\|_2$, and
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g\, dx = \int_{\mathbb{R}} fg\, dx \qquad \text{for all } \ g \in L^2.$$
Prove that $f_n \to f$ in $L^2(\mathbb{R})$ norm.

7. Suppose f is an $L^2(\mathbb{R})$ function such that the function $g(x) = xf(x)$ also belongs to $L^2(\mathbb{R})$. Prove that $f \in L^1(\mathbb{R})$ and

$$\|f\|_1 \leq \sqrt{2}\,(\|f\|_2 + \|g\|_2).$$