Joint Program Exam in Real Analysis

September 2011

Instructions:

1. You may use up to three and a half hours to complete this exam.
2. The exam consists of 7 problems. You need to do ALL of them for full credit.
3. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.
4. Throughout the exam m denotes the Lebesgue measure. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. $L_p(a,b)$ denotes the L_p space with respect to m on the interval (a,b), $-\infty \leq a < b \leq \infty$.

1. Are the following statements true or false? Justify! If a statement is false, provide a counter-example.

   a) If $f: [0,1] \to \mathbb{R}$ is continuous a.e., then f is measurable.

   b) If the set $\{x \in [0,1]: f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable.

   c) The characteristic function of the Cantor set is Lebesgue integrable in $[0,1]$ but not Riemann integrable.

2. Let $f \in L_1(\mathbb{R})$. Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 0.$$

3. Given that $\int_0^\infty e^{-x} \sin(x)\, dx = \frac{1}{2}$, prove that

$$\int_0^\infty e^{-x} \sqrt{3 + 2\sin(x)}\, dx \le 2.$$

4. Let $\varepsilon > 0$ and

$$f(x) = \begin{cases} x^{2+\varepsilon} \sin \frac{1}{x^2}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

   Prove that f is absolutely continuous on $[0,1]$.

5. Does there exist an absolutely continuous function $f: [0,1] \to \mathbb{R}$ and a sequence of Borel sets $E_n \subset [0,1]$, $n = 1, 2, \ldots$, such that $m(E_n) > 0$ and

$$\frac{m(f(E_n))}{m(E_n)} > n$$

   for all $n \ge 1$?

6. Let f be a measurable function on $\mathbb{R}$. Prove that

$$\int_{\mathbb{R}} |f|^2 \, dm = 2 \int_0^\infty t \, m(\{x : |f(x)| > t\}) \, dt.$$

(Hint: Use Fubini's Theorem.)

7. Let $K \in L_1(\mathbb{R})$ and f be measurable, bounded on $\mathbb{R}$, and $\lim_{x \to \infty} f(x) = 0$. Let

$$F(x) := \int_{\mathbb{R}} K(x - s) f(s) \, ds \qquad (x \in \mathbb{R}).$$

Prove that $F(x)$ is finite for all $x \in \mathbb{R}$ and

$$\lim_{x \to \infty} F(x) = 0.$$