Joint Program Exam, September 2010

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this exam m and m^* denote Lebesgue measure and Lebesgue outer measure, respectively. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. $L^p(\mathbb{R})$ and $L^p(a,b)$ denote the L^p space with respect to m on $\mathbb{R}$ and the interval (a,b), respectively.

Part 1.

DO ALL PROBLEMS IN PART ONE.

Are the following statements true or false? Justify! If a statement is false, provide a counterexample.

1. Let E be a subset of $\mathbb{R}^n$, and $\operatorname{int} E$ the set of all interior points of E. Then $\operatorname{int} E = \emptyset$ if and only if $m^*(E) = 0$.

2. Suppose that f is continuous and Lebesgue integrable on $\mathbb{R}$. Then $\lim_{x\to\infty} f(x) = 0$.

3. Suppose that $f_n \in L^1(0,1)$ for all $n \in \mathbb{N}$, that $f_n(x) \to g(x)$ for almost every $x \in [0,1]$ and that $f_n \to f$ in $L^1(0,1)$. Then $f(x) = g(x)$ for almost every $x \in [0,1]$.

4. Let f be a continuous real-valued function on $[0,1]$. Suppose that
$$|f'(x)| \leq 3 \quad \text{for all } x \in [0,1].$$
Then f is absolutely continuous on $[0,1]$.

5. If $f : [0,\infty) \to \mathbb{R}$ is continuous and the improper Riemann-integral
$$\lim_{R\to\infty} \int_0^R f(x)\, dx$$
exists and is finite, then $f \in L^1(0,\infty)$.

Part 2.

DO 5 OF THE 6 PROBLEMS IN PART TWO. IF YOU WORK ON ALL PROBLEMS, MARK THE ONES TO BE GRADED.

1. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that for every open set $\mathcal{O}$ in $\mathbb{R}$,
$$f^{-1}[\mathcal{O}] \quad \text{is a Borel set in } \mathbb{R}.$$
Show that for every Borel set $\mathcal{B}$ in $\mathbb{R}$,
$$f^{-1}[\mathcal{B}] \quad \text{is a Borel set in } \mathbb{R}.$$

2. For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ let $f_{n,k}$ be non-negative and measurable on $\mathbb{R}$ and assume that $\int_{\mathbb{R}} f_{n,k}\, dm \leq \frac{1}{n^2}$. Show that
$$f := \sum_{n=1}^{\infty} \liminf_{k\to\infty} f_{n,k} \in L^1(\mathbb{R}).$$
Here the series and liminf are defined pointwise.

3. Find
$$\lim_{k\to\infty} \int_0^{\infty} \frac{dx}{(1+\frac{x}{k})^k \cdot \sqrt[k]{x}}.$$
Justify your answer!

4. Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for all $n \in \mathbb{N}$. Prove or disprove:
(a) If $\|f_n\|_1 \to 0$, then $\|f_n\|_2 \to 0$.
(b) If $\|f_n\|_2 \to 0$, then $\|f_n\|_1 \to 0$.

5. Let f and g be in $L^2(\mathbb{R})$ and define
$$h(x) := \int_{\mathbb{R}} f(x-y)g(y)\, dy$$
for all $x \in \mathbb{R}$. Show that h is bounded and continuous on $\mathbb{R}$.

6. Let f be a positive measurable function defined on a measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$. Prove that
$$\left(\int_E f\, dm\right)\left(\int_E \frac{1}{f}\, dm\right) \geq m^2(E),$$
and that equality in the above inequality holds if and only if $f(x) = c$ almost everywhere in E for some constant $c > 0$.