Joint Program Exam, May 2010

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this exam m denotes Lebesgue measure on $\mathbb{R}$ and m_n Lebesgue-measure on $\mathbb{R}^n$. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dt, referring to the variable to be integrated. $L^p(\mathbb{R})$ and $L^p(a, b)$ denote the L^p space with respect to m on $\mathbb{R}$ and the interval (a, b), respectively.

Part 1.

DO ALL PROBLEMS IN PART ONE.

Are the following statements true or false? Justify! If a statement is false, provide a counterexample.

1. Let $\{f_k\}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_k \to f$ a.e. in $\mathbb{R}$. Then $\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, dm$ exists and
$$\int_{\mathbb{R}} f \, dm \leq \lim_{k\to\infty} \int_{\mathbb{R}} f_k \, dm.$$
2. If F is a proper closed subset of $[0,1]^n$, then $m_n(F) < 1$.
3. We have
$$\lim_{n\to\infty} \int_0^1 e^{x^2/n} \, dx = \int_0^1 \lim_{n\to\infty} e^{x^2/n} \, dx.$$
4. The two iterated integrals below are equal:
$$\int_0^\infty \left(\int_{-\infty}^\infty e^{-tx^2} \, dx \right) dt = \int_{-\infty}^\infty \left(\int_0^\infty e^{-tx^2} \, dt \right) dx.$$
5. If f is continuous on $[0,1]$ and f is of bounded variation on $[0,1]$, then f is absolutely continuous on $[0,1]$.

Part 2.

DO 5 OF THE 6 PROBLEMS IN PART TWO. IF YOU WORK ON ALL PROBLEMS, MARK THE ONES TO BE GRADED.

1. (a) Show that
$$L^3(\mathbb{R}) \cap L^6(\mathbb{R}) \subseteq L^4(\mathbb{R}).$$
(b) Show that $L^3(\mathbb{R}) \cap L^6(\mathbb{R})$ is not contained in $L^2(\mathbb{R})$.

2. If $f \in L^1(0,1)$, find
$$\lim_{k\to\infty} \int_0^1 k \ln\left(1 + \frac{|f(x)|^2}{k^2}\right) dx.$$
Justify your answer.

3. (a) Let f be measurable on $[0,1]$. For $1 \leq p < \infty$ define
$$g(p) = \left(\int_{[0,1]} |f|^p \, dm\right)^{1/p}.$$
Show that g is non-decreasing on $[1,\infty)$.

   (b) Assume in addition that $f \notin L^\infty(0,1)$. Show that $\lim_{p\to\infty} g(p) = \infty$.

4. Let $E \subset \mathbb{R}$ be a Borel set such that $0 < m(E) < \infty$. Let $f \geq 0$ be a Borel-measurable function on $\mathbb{R}$ and define
$$g(x) = \int_E f(x-t)\, dt, \quad x \in \mathbb{R}.$$
Prove that g is measurable on $\mathbb{R}$ and that $f \in L^1(\mathbb{R})$ if and only if $g \in L^1(\mathbb{R})$.

5. Let $\mathcal{M}$ and $\mathcal{N}$ denote σ-algebras on the sets X and Y, respectively. Define $\mathcal{M} \times \mathcal{N}$ as the smallest σ-algebra on $X \times Y$ containing all sets of the form $A \times B$, where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. For $E \subseteq X \times Y$ and $y \in Y$, define $E^y = \{x \in X : (x,y) \in E\}$.

   Show that for any $E \in \mathcal{M} \times \mathcal{N}$ and any $y \in Y$, we have $E^y \in \mathcal{M}$.

6. (a) Let $f : [a, b] \to \mathbb{R}$ be strictly increasing and continuous and let $E \subseteq [a, b]$ be a Borel set. Show that $f(E) = \{f(x) : x \in E\}$ is a Borel set.

(b) Assume in addition that f is absolutely continuous on $[a, b]$. Show that

$$m(f(E)) = \int_E f' \, dm.$$