Joint Program Exam in Real Analysis

May 2009

Instructions:

(1) You may take up to three and a half hours to complete the exam.
(2) Completeness in your answers is very important. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".
(3) Justify all your claims. When appropriate, refer to a theorem by name or by providing a complete statement of its content.

Notation:
$\mathbb{R}$ denotes the set of real numbers. m denotes Lebesgue measure and m^* the associated outer measure. Sometimes we use $d\mathrm{m}(x)$ or dx or the like to indicate which variable is the variable of integration. All integrals occurring are Lebesgue integrals.

(1) Define $\mu(\{n\}) = 2^{-n}$ for every $n \in \mathbb{N}$.
 (a) Determine the largest σ-algebra $\mathcal{M}$ in $\mathbb{N}$ such that μ extends to a measure on $\mathcal{M}$.
 (b) Compute $\int_{\mathbb{N}} \mathrm{e}^{-x} d\mu$.

(2) Let $f \in L^2([0,1])$ and define

$$g(x) = x^{-4/3} \int_0^x f(t)\,dt, \qquad 0 \le x \le 1.$$

Show that $\|g\|_1 \le 6\|f\|_2$.

(3) For $f \in L^1([0,\infty))$ and for $x \ge 0$, define

$$F(x) = \int_{(x,\infty)} f(t)\mathrm{e}^{x-t}\,\mathrm{dm}(t).$$

Show that $F \in L^1([0,\infty))$.

(4) Let $f \in L^1([0,1])$ be real-valued. Prove the following statements:
 (a) $x^k f(x) \in L^1([0,1])$ for all $k \in \mathbb{N}$.
 (b) $\lim_{k\to\infty} \int_0^1 x^k f(x)\,dx = 0$.
 (c) If $\lim_{x\uparrow 1} f(x) = a$ for some real number a, then

$$\lim_{k\to\infty} k \int_0^1 x^k f(x)\,dx = a.$$

(5) Suppose that f is absolutely continuous on $[a,b]$ and that $f' \in L^p([a,b])$ for some $p \in (1,\infty)$. Show that f satisfies a Hölder condition for order α for some $\alpha \in (0,1]$, i.e., show that

$$\sup\left\{\frac{|f(s)-f(t)|}{|s-t|^\alpha} : s,t \in [a,b], s \ne t\right\} < \infty.$$

Which is the biggest (and hence best) α which works for any such function f?

(6) Suppose that f_n is a sequence of nonnegative Lebesgue measurable functions on $(0,10)$ and that $f_n(x) \to f(x)$ for almost all $x \in (0,10)$. Let $F(x) = \int_0^x f\mathrm{dm}$ and $F_n(x) = \int_0^x f_n\mathrm{dm}$. Prove that

$$\int_0^{10} (f+F)\mathrm{dm} \le \liminf_{n\to\infty} \int_0^{10} (f_n+F_n)\mathrm{dm}.$$

(7) Let $(X, \mathcal{M}, \mu)$ be a measure space with a positive, finite measure μ. Consider a function $f \in L^\infty(\mu)$ such that $\|f\|_\infty > 0$.

(a) Show that, for every positive ε, the set $\{x : |f(x)| > \|f\|_\infty - \varepsilon\}$ has positive measure.

(b) Show that

$$\lim_{n\to\infty} \|f\|_n = \lim_{n\to\infty} \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} = \|f\|_\infty.$$

(8) (a) Assume that $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Show that f^2 is absolutely continuous on $[a, b]$.

(b) Assume that $h \in L^1(a, b)$. Prove that there exists a unique $g \in L^1(a, b)$ such that

$$\int_a^x g(t)\, dt = \left(\int_a^x h(t)\, dt \right)^2$$

for all $x \in [a, b]$.