Joint Program Exam in Real Analysis

Spring 2008

Instructions

1. You may take up to $3\frac{1}{2}$ hours to complete the exam.
2. Work seven out of the eight problems. Completeness in your answers is very important. An essentially complete and correct solution to one problem with gain more credit than solutions to two problems, each of which is "half correct".
3. All of the numbered problems have equal weight.
4. When appropriate, refer to a theorem by name or by providing a brief theorem statement. You do not need to reprove theorems you used in your proof.

Notation

Throughout the exam, $\mathbb{R}$ stands for the set of real numbers. Lebesgue measure on $\mathbb{R}$ is denoted by m and on $\mathbb{R}^2$ by m_2. On the real line, notation such as $\int_{[0,1]} f$, $\int_{[0,1]} f\,dm$, $\int_{[0,1]} f(x)dx$, etc. is used for Lebesgue integrals, while Riemann integrals are denoted $\int_0^1 f(x)dx$, $\int_0^\infty f(x)dx$, etc.

1. Prove or disprove each of the following statements.

   (a) There exists a sequence (f_n) of functions in $L^1[0,\infty)$ such that $|f_n(x)| \leq 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and

   $$\lim_{n\to\infty} \int_{[0,\infty)} f_n = 1$$

   (b) There exists a sequence (g_n) of functions on $[0,1]$ such that

   $$\lim_{n\to\infty} \int_{[0,1]} g_n = 0$$

   but $g_n(x)$ converges for no $x \in [0,1]$.

   Note. Be brief but convincing!

2. Are the following statements true or false? Justify! (Briefly but convincingly.)

   (a) For every measurable subset Ω of the square $\{0 < x, y < 1\}$, the Lebesgue measure $m_2(\Omega) = \sup \sum_i |P_i|$, where the supremum is taken over all such countable collections of disjoint rectangles $\{P_i\}$ that $\bigcup_i P_i \subseteq \Omega$, and $|P_i|$ is the area of P_i.

   (b) For every non-negative, bounded and measurable function f on $[0,1]$, $\int_{[0,1]} f\, dm = \inf \int_{[0,1]} \varphi\, dm$, where the infimum is taken over all simple measurable functions φ with $f \leq \varphi$.

3. Find $\lim_{n\to\infty} \int_{[0,\pi/2]} \sqrt{n \sin\left(\frac{x}{n}\right)}\, dm$.

4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = \begin{cases} e^{-x^2-y^2} & \text{if } x^2 - y \text{ is irrational} \\ 1 & \text{otherwise} \end{cases}$.

   Compute $\int_{\mathbb{R}^2} g\, dm_2$.

5. Let A be the class of absolutely continuous real valued functions on $[0,4]$ such that $f(0) = 0$ and $\int_{[0,4]} [f'(x)]^2 dx = 1$. Find (and justify!)

   (a) $\sup_{f\in A} |f(4)|$

   (b) $\inf_{f\in A} |f(4)|$

6. Are the following statements about total variation true or false? Justify!

   (a) For every function of bounded variation on $[a,b]$, $V_a^b(f) \leq V_a^b(|f|)$.

   (b) For every function of bounded variation on $[a,b]$, $V_a^b(|f|) \leq V_a^b(f)$.

   (c) For every function of bounded variation on $[a,b]$, $V_a^b(f) = V_a^b(|f|)$.

7. Let $\epsilon > 0$ and $f(x) = \begin{cases} x^{2+\epsilon} \cos \frac{1}{x^2} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$. Prove: f is absolutely continuous on $[0, 1]$.

8. Show that $\left(\int_0^1 \frac{x^3}{(1-x)^{1/5}} dx \right)^5 \leq \frac{16}{81}$.