JOINT PROGRAM EXAM, September 2006

REAL ANALYSIS

Instructions:

You may use up to $3\frac{1}{2}$ hours to complete this exam.

On each page of your solutions, write down your student ID number, and the number of the problem being answered.

Justify the steps in your solutions by referring to theorems by name when appropriate, and by verifying the hypothesis of these theorems. You do not need to reprove the theorems you used.

For each problem you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit then two "half solutions" to two problems.

Notations:

R denotes the set of real numbers, $m(A)$ refers to the Lebesgue measure of set $A \subset R^d$, "a.e." means almost everywhere with respect to Lebesgue measure. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. The symbol $\mu(A)$, $A \subset X$, refers to a general measure on a measure space X. If you are accustomed to different notations, feel free to use those (after describing them).

PART I.

Do all the problems in Part I.

1. Find the Lebesgue integral $\int_{[0,1]\times[0,1]} h(x,y)dm$ where

$$h(x,y) = \begin{cases} 1, & \text{if } xy \text{ is irrational} \\ 0, & \text{if } xy \text{ is rational.} \end{cases}$$

2. Denote $I = [0,1]$. Let $f : I \times I \to R$ be measurable and such that

$$\int_I \left(\int_I f(x,y)dx \right) dy = 1, \qquad \int_I \left(\int_I f(x,y)dy \right) dx = -1.$$

Find the range of values of $\int_{I\times I} |f(x,y)|\, dm$ over all such functions f. Justify your answer.

3. Prove or disprove: any sequence of measurable functions on [0,1] that converges to zero a.e. on $[0,1]$ must converge to zero in $L^1[0,1]$.

4. Suppose $f : [0,1] \to R$ is a function of bounded variation. Show that $g : [0,1] \to R$ defined by $g(x) = f(x^\beta)$, $\beta > 0$, is also a function of bounded variation.

PART II.

Do 4 out of 5 problems in Part II.

1. Let $\mu(X) < \infty$ and f be a measurable function on X. Suppose there are constants $A > 0$ and $\alpha > 2$ such that
$$\mu\{x \in X : |f(x)| > M\} < AM^{-\alpha}$$
for any $M > 0$. Prove that $f(x)$ is integrable.

2. Find the limit:
$$\lim_{n\to\infty} \int_0^\infty \left(\frac{\sin x}{x}\right)^n dx.$$
Justify all the steps.

3. Show that absolutely continuous function $f : [a, b] \to R$ transforms any set E of Lebesgue measure zero into the set $f(E)$ of measure zero.

4. Suppose that $A \subset R$ is a Lebesgue measurable set for which there is a constant c, $0 \le c < 1$, such that $m(A \cap I) \le cm(I)$ for every interval I. Prove that $m(A) = 0$.

5. Let $f(x)$ be a nonnegative continuous function on $[0, 1]$. Prove that the Riemann integral $\int_0^1 f(x)dx$ is equal to the Lebesgue integral $\int_{[0,1]} f(x)dx$.