Joint Program Exam in Real Analysis

September 10, 2002

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. *Do seven problems out of eight*. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems, each of which is "half correct".

Notation: Throughout the exam, "R" stands for the set of real numbers. Notation such as $\int_{[1,0]} f$, $\int_{[1,0]} f(x)\,dx$, etc. is used for Lebesgue integral, while Riemann integral is denoted $\int_0^1 f(x)\,dx$, $\int_0^\infty f(x)\,dx$, etc.

1. Let f: [0, 1] $\to$ R be monotone increasing, that is $f(y) \geq f(x)$ whenever $y > x$. Show that f has at most countably many discontinuities.

2. Let f, f_k be integrable on [0, 1], k = 1, 2, ? . Suppose that $f_k \to f$ a.e. and $\int_{[0,1]} |f_k| \to \int_{[0,1]} |f|$. Prove that $\int_{[0,1]} |f_k - f| \to 0$.

3. Find the limit (justify steps): $\lim_{n\to\infty} \int_0^1 \frac{(nx)^2}{(1+x^2)^n}\, dx$.

4. Let f: [0, 1] $\to$ R be Lebesgue−measurable and non−negative, and let m denote one−dimensional Lebesgue measure.

   (a) Show that

$$\int_{[0,1]} f\, dm = \int_0^\infty m(\{x \in [0,1] : f(x) > t\})\, dt.$$

   (b) Suppose in addition that there exists a finite constant C such that

$$m(\{x \in [0,1] : f(x) > t\}) \leq \frac{C}{t}$$

   for all $t > 0$. Show that $f^{\,s} \in L^1([0,1])$ for all $s \in (0,1)$.

5. Let $f(x,y) \geq 0$ be measurable on $\mathrm{R}^n \times \mathrm{R}^n$. Suppose that, for a.e. $x \in \mathrm{R}^n$, $f(x,y)$ is finite for a.e. y. Prove that, for a.e. $y \in \mathrm{R}^n$, $f(x,y)$ is finite for a.e. x.

6. Denote $I = [0, 1]$. Let f: $I \times I \to \mathrm{R}$ be measurable and such that $\int_I \left[\int_I f(x,y)\, dy\right] dx = 1$ and $\int_I \left[\int_I f(x,y)\, dx\right] dy = -1$. Find the range of values of $\iint_{I\times I} |f(x,y)|\, dxdy$ over all such functions f.

7. Show that

$$\left(\int_0^1 \frac{x^{1/2}\, dx}{(1-x)^{1/3}}\right)^3 \leq \frac{8}{5}.$$

8. Let $g \in L^1(\mathrm{R})$ and $G(x) = \int_{\mathrm{R}} g(y)\, e^{-(x-y)^2}\, dy$. Prove that, for any $p \in [1,\infty)$,

$G \in L^p(\mathbb{R})$ and estimate $\|G\|_p$ in terms of $\|g\|_1$.