Joint Program Exam, May 2001

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test m_n denotes Lebesgue measure on $\mathbb{R}^n$ and 'measurable' is short for 'Lebesgue-measurable'. Instead of dm_1 we write dx.

Part 1.

DO ALL PROBLEMS IN PART ONE.

1. (a) Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable? Justify.
(b) Does there exist a non-measurable subset of $\mathbb{R}$ whose complement in $\mathbb{R}$ has outer measure zero? Justify.
(c) Do there exist two non-measurable sets whose union is measurable? Justify.

2. (a) Let $p > q \geq 1$. Show by example that $L^p([0,1]) \neq L^q([0,1])$.
(b) Show by example that there exist two functions $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ such that $f + g$ is neither in $L^1(\mathbb{R})$ nor in $L^2(\mathbb{R})$.

3. Let f and f_n, $n = 1, 2, \ldots$, be non-negative measurable functions on $[0,1]$ such that f_n converges pointwise to f. Under each of the following additional assumptions, either prove that $\int_0^1 f_n \, dm \to \int_0^1 f \, dm$ or show that this is not generally true. Integrals and convergence are to be understood in the sense of extended real numbers.
(a) $f_n \geq f$ and $f_n \in L^1([0,1])$ for all n,
(b) $f_n \geq f_{n+1}$ for all n,
(c) $f_n \leq f$ for all n.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. (a) Let $a_1, \dots, a_n$ be positive numbers. Prove that their harmonic mean is bounded by their arithmetic mean, i.e.

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^{-1} \leq \frac{1}{n}\sum_{k=1}^{n} a_k.$$

(b) Characterize the vectors $(a_1, \dots, a_k)$ for which equality holds in (a).

2. Compute

$$\lim_{n\to\infty}\int_0^1 \frac{nx^{n-1}}{2+x}\,dx.$$

Make sure that you justify your answer with appropriate convergence theorems.

3. Let $E \in \mathbb{R}^n$ be measurable with $m_n(E) > 0$. Show that $L^2(E) \not\subset L^\infty(E)$.

4. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with bounded support, i.e. $\{x : f(x) \neq 0\}$ is bounded, and let $g \in L^1(\mathbb{R})$. Define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy \qquad (x \in \mathbb{R}).$$

(a) Show that $f * g$ is continuous.
(b) If, in addition, f is continuously differentiable, then prove that $f * g$ is continuously differentiable and $(f * g)'(x) = (f' * g)(x)$.

5. Suppose $f \in L^1(\mathbb{R}^2)$ is real-valued. Show that there exists a measurable set $E \subset \mathbb{R}^2$ such that

$$\int_E f dm_2 = \int_{\mathbb{R}^2\setminus E} f dm_2.$$

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and let

$$E = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

Show that $m_2(E) = 0$.