# Joint Program Examination
# Real Analysis
# Fall 1995

## Time: Three and One Half Hours

**Instructions:** Completeness is your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct answer to one problem will gain more credit than solutions to two problem, each of which is only "half-correct".

Full credit will be awarded for answering correctly any eight of the ten problems given.

**Notation:** $\mathbb{R}$ denotes the set of real numbers, $m(E)$ refers to the Lebesgue measure of the set $E \subset \mathbb{R}$, and "a.e." means almost everywhere with respect Lebesgue measure.

**Question 1** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative functions on $[0,1]$ which converge pointwise to $f$, and suppose that $f_n \le f$ for each $n$. Prove that

$$\int f\,dm = \lim_n \int f_n\,dm.$$

**Question 2** Let $f \in L^1(\mathbb{R})$, and define

$$g(x) = \int_{-\infty}^{x} f\,dm.$$

Prove that $g$ is continuous.

**Question 3** Evaluate:

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} \left(1 - \sqrt{\sin x}\right)^n \cos x\,dx.$$

**Question 4** Evaluate:

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)}\,dx.$$

**Question 5** Let $\mathbb{N}$ denote the positive integers, and let $\alpha > 2$ be a real number. Define

$$E = \{x \in [0,1] \colon\ |x - p/q| < q^{-\alpha} \text{ for infinitely many pairs } (p,q) \in \mathbb{N}^2\}.$$

Prove that $m(E) = 0$.

**Question 6** Let $E$ be a subset of $\mathbb{R}$, and let $0 < \alpha < 1$. Prove:

(a) If $m(E \cap I) \ge \alpha m(I)$ for all open intervals $I \subset \mathbb{R}$, then $m(E) = \infty$.

(b) If $m(E \cap I) \le \alpha m(I)$ for all open intervals $I \subset \mathbb{R}$, then $m(E) = 0$.

**Question 7** Let $f \in L^1([0,1])$ and suppose $f$ is nonnegative. Show that

$$\int_0^1 \frac{f(y)}{|x-y|^{1/2}}\,dy$$

is finite for a.e. $x \in [0,1]$.

**Question 8** Suppose that $f$ is a nonnegative and absolutely continuous function on $[0,1]$. Prove or disprove: $\sqrt{f}$ is absolutely continuous.

**Question 9** Let $f$ and $g$ be nonnegative functions on $[0,1]$ satisfying

$$fg \geq 1.$$

Prove that

$$\int_0^1 f\,dm \cdot \int_0^1 g\,dm \geq 1.$$

**Question 10** Let $f \in L^\infty([0,1])$. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.