Joint Program Exam in Mathematical Analysis
May 6, 2024

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, so that $\lim_n a_n = L$. Prove that

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \dots a_n} = L.$$

Make sure that you consider the possibility that $L = 0$.

2. Let (X, d) be a metric space, and $K_1, K_2, \dots K_n, \dots$ be an infinite sequence of compact subsets of X. Prove that if $\cap_{n=1}^{\infty} K_n = \emptyset$, then there exists N, so that $\cap_{n=1}^{N} K_n = \emptyset$.

3. Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1]$ such that x_n are distinct and $\lim_{n\to\infty} x_n = 0$. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin n, & \text{if } x = x_n, \\ 1, & \text{if } x \notin X. \end{cases}$$

Show that f is Riemann integrable on $[0, 1]$. Compute $\int_0^1 f(x)dx$.

4. Suppose that $f : [0, +\infty) \to \mathbb{R}$ is a bounded and locally integrable function with $\lim_{x\to+\infty} f(x) = L$. Prove that

$$\lim_{t\to 0+} t \int_0^{\infty} e^{-tx} f(x)dx = L.$$

5. (i) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n + x^2}$$

converges uniformly on $\mathbb{R}$, but it is not absolutely convergent for any $x \in \mathbb{R}$.

(ii) Let $f(x)$ be the sum function of the above series for $x \in \mathbb{R}$. Show that f is continuously differentiable on $\mathbb{R}$.

6. Let $f : [0, 1] \to \mathbb{R}$ be a Lipschitz function, i.e. $|f(x) - f(y)| \le L|x - y|$, and hence Riemann integrable. Prove that

$$\left| \int_0^1 f(x)dx - \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{k}{N}\right) \right| \le \frac{L}{2N}.$$

Hint:

$$\int_0^1 f(x)dx = \sum_{k=1}^{N} \int_{\frac{k-1}{N}}^{\frac{k}{N}} f(x)dx$$

7. Let $\{a_n\}$ be a real sequence, with $a_n \neq 0$. Define

$$\begin{aligned} \lambda_0 &= \inf\{\lambda : \text{There exists } C \text{ so that} |a_n| \le Ce^{n\lambda}\} \\ \mu_0 &= \limsup_n \frac{\ln |a_n|}{n} \end{aligned}$$

Show that $\lambda_0 = \mu_0$. You may assume that both λ_0, μ_0 are finite.