Joint Program Exam in Real Analysis

May 3, 2022

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}_d(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of m_1 we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\|\cdot\|_p$ denotes the norm on $L^p(X, \mu)$. We may also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when I is a subinterval of $\mathbb{R}$.

1. Given any number in $[0,1]$ form its decimal expansion without repeating 9's at the end (for example, for $1/2$ we take .5 and not $.4999\ldots$). If the decimal expansion of a number $x \in [0,1]$ has no 5 in it, we say that x is in S. Prove that S is Lebesgue measurable and find its measure.

2. Find
$$\lim_{n\to\infty} \int_0^\infty \frac{nx^{1/n}}{3ne^x + \cos(nx)}\, dx.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space with a positive measure μ and suppose $f : X \to [1, \infty)$ is a measurable function. Determine
$$\lim_{n\to\infty} \int_X \frac{f^n}{1+f^n}\, d\mu.$$

4. Prove that $\lim_{t\to\infty} \int_1^t \frac{\sin x}{x}\, dx$ exists. Does $\frac{\sin x}{x}$ belong to $L^1([1,\infty))$? Justify your answer.

5. Fix $\varepsilon > 0$ and define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} x^{2+\varepsilon}\cos(1/x^2) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$
Prove that f is absolutely continuous.

6. Let $f \in L^\infty([0,1])$ and, for $n \in \mathbb{N}$,
$$a_n = \int_{[0,1]} |f|^n\, dx.$$
Assume $a_1 \neq 0$ and show that $\lim_{n\to\infty} a_{n+1}/a_n = \|f\|_\infty$.

7. Suppose $f \in L^1(\mathbb{R})$. Show that
$$h(x) = \int_{(0,1)} \frac{f(x-t)}{\sqrt{t}}\, dt$$
is defined for almost every $x \in \mathbb{R}$ and that $\|h\|_1 \leq 2\|f\|_1$.

8. Find all nonnegative functions $g \in L^{3/2}((0,2))$ such that
$$\left(\int_0^2 xg(x)\, dx\right)^6 = 16\left(\int_0^2 g(x)^{3/2}\, dx\right)^4.$$