Joint Program Exam in Mathematical Analysis
September 12, 2023

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.
2. You may use up to three and a half hours to complete this exam.
3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.
4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.
5. None of the problems require the use of results from measure theory or complex analysis. You cannot invoke results about measurable functions or analytic functions unless you prove them (such results are not needed to solve the problems). If you are in doubt about a statement ask the faculty proctoring the exam.

1. Assume that for a positive sequence $\{a_n\}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a > 0$. Prove that $\lim_{n\to\infty} a_n^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = a.$$

2. Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$, x_n are distinct and $\lim_n a_n = 0$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} a_n & \text{if } x = x_n, \\ 0 & x \notin X. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$. Compute $\int_0^1 f(x)dx$.

3. Let (X,d) be a metric space. Recall that for a set $A \subset X$ and $x \in X$, $d(x,A) := \inf\{d(x,y) : y \in A\}$. Also

$$\bar{A} = \cap\{F : F \supset A, \quad F - \text{closed}\}.$$

Prove that

$$\bar{A} = \{x \in X : d(x,A) = 0\}.$$

4. Let $f_n : [0,1] \to \mathbb{R}, n = 1,2,\ldots$ and $f : [0,1] \to \mathbb{R}$ be continuous functions, so that

1. For each $x \in [0,1]$ $f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots \leq f(x)$.
2. For each $x \in [0,1]$, $f(x) = \lim_n f_n(x)$ (that is $f_n \to f$).

Prove that f_n converges uniformly to f on $[0,1]$.
Hint: Argue by contradiction.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, with $f'(a) = f'(b) = 0$. Prove that there exists $\xi \in (a,b)$, so that

$$\frac{4}{(b-a)^2}|f(b) - f(a)| \leq |f''(\xi)|.$$

Hint: Apply the Taylor's theorem twice, to evaluate $f(\frac{a+b}{2})$.

6. Let $f, g : [0,\infty) \to \mathbb{R}$ be locally integrable functions. That is f, g are Riemann integrable in each interval $[0,N]$, for all $N > 0$. In addition, $\int_0^{\infty} |g(x)|dx < \infty$, $\lim_{x\to\infty} f(x) = 0$. Prove that the function

$$h(x) = \int_0^x f(x-y)g(y)dy, \qquad \forall\, x \in [0,\infty),$$

has the property $\lim_{x\to\infty} h(x) = 0$.

7. Let $L > 0$. Let $\{f_a\}_{\mathcal{A}}$ be a collection of real-valued functions on $\mathbb{R}$, with the property that

$$|f_a(x) - f_a(y)| \leq L|x-y|, \quad \forall a \in A, x, y \in \mathbb{R},$$

that is, each f_a is L-Lipschitz continuous. Suppose that there exists $x_0 \in \mathbb{R}$ with

$$\inf_{a\in\mathcal{A}} f_a(x_0) > -\infty.$$

Show that

- the function $g(x) := \inf_{a\in\mathcal{A}} f_a(x)$ is well-defined, i.e. $g(x) > -\infty$ for each x,
- g is L-Lipschitz continuous, that is, $|g(x) - g(y)| \leq L|x-y|, \quad \forall x, y \in \mathbb{R}$.