Joint Program Exam in Real Analysis
September 6, 2022

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.
2. You may use up to three and a half hours to complete this exam.
3. The exam consists of 8 problems. All problems are weighted equally.
4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.
5. Except in Problem 3, all integrals throughout the exam mean the Lebesgue integrals, $L^p(E)$ denotes the L^p space with respect to the Lebesgue measure on the Lebesgue measurable set E, and "measurable" refers to Lebesgue measurable. $\mathbb{R}$ denotes the set of real numbers.

1. Determine whether each of the following statements is true or false? Justify your answers.

   (a) If $f \in L^1([0,\infty))$ and $f \geq 0$ then $\lim_{x\to\infty} f(x) = 0$.

   (b) If $\lim_{x\to\infty} f(x) = 0$ and $f \geq 0$ on $[0,\infty)$ then $f \in L^1([0,\infty))$.

   (c) If $f \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} |xf(x)|\, dx < \infty$, then $f \in L^1(\mathbb{R})$.

2. Suppose $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N xf(x)\, dx = 0.$$

3. Suppose $(X, \mathcal{M}, \mu)$ is a measure space with a positive measure and that $f : X \to (0,\infty)$ satisfies $\int_X f d\mu = 1$. Prove that if $0 < \mu(E) < \infty$ and $0 < r < 1$, then

$$\int_E \log f d\mu \leq -\mu(E) \log(\mu(E)),$$

and

$$\int_E f^r d\mu \leq \mu(E)^{1-r}.$$

4. Define the sequence (f_n) of functions on $\mathbb{R}$ by $f_n = \chi_{[n,n+1]}$ for $n = 1, 2, \cdots$, where $\chi_{[n,n+1]}$ is the characteristic function of the interval $[n, n+1]$.

   (i) Show that for $1 < p < \infty$ and any function $g(x) \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$, $\int_{\mathbb{R}} f_n g\, dx \to 0$ as $n \to \infty$.

   (ii) Is it true that for any $g \in L^\infty(\mathbb{R})$, $\int_{\mathbb{R}} f_n g\, dx \to 0$ as $n \to \infty$? Justify your answer.

5. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Show that f is absolutely continuous on $[0,1]$.

6. Let $f \in L^1(\mathbb{R})$ and $\alpha \in (0,1)$. Show that for almost every $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{|f(x)|}{|x-y|^\alpha}\, dx < \infty.$$

7. Let (f_n) be a sequence in $L^2((0,1))$ such that $\int_0^1 |f_n|^2\, dx \leq 1$ and $f_n \to f$ in measure as $n \to \infty$. Prove that

   (i) $\int_0^1 |f|^2\, dx \leq 1$;

   (Hint: The following Riesz theorem can be directly applied: Any sequence convergent in measure on a measurable set E has a subsequence that converges almost everywhere on E.)

   (ii) $\int_0^1 |f_n - f|\, dx \to 0$ as $n \to \infty$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying

   (a) for almost every $t \in \mathbb{R}$, $f(t,x)$ is continuous in x;

   (b) for every $x \in \mathbb{R}$, $f(t,x)$ is Lebesgue measurable in t.

   Show the following:

   (i) If $\chi_E : \mathbb{R} \to \mathbb{R}$ is the characteristic function of a Lebesgue measurable set $E \subset \mathbb{R}$, then $f(t, \chi_E(t))$ is Lebesgue measurable on $\mathbb{R}$.

   (ii) If $\phi : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then $f(t, \phi(t))$ is Lebesgue measurable on $\mathbb{R}$.