UNIVERSITY OF ALABAMA SYSTEM JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS JOINT PROGRAM EXAMINATION Linear Algebra / Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 1994

Instructions: Do only 7 of the 8 problems given. Be sure to indicate which 7 should be graded. Include all work for full credit.

1. Let $\mathbf{P}_3(\mathbf{R})$ be the space of all polynomials over the real line, of degree less than or equal to 3. Define $T : \mathbf{P}_3(\mathbf{R}) \to \mathbf{P}_3(\mathbf{R})$ by

$$T(p) = tp' + p$$

where the superscript denotes differentiation with respect to t. Show that T is a linear transformation and find a matrix representation with respect to a basis of your choice. Prove or disprove:

   (a) T is 1-1;

   (b) T is onto.

2. Let $\mathbf{A} \in \mathbf{R}^{m \times n}$, rank $\mathbf{A} = n \leq$ m. Prove that there exists $\mathbf{B} \in \mathbf{R}^{n \times m}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{B}\mathbf{b}$. Is $\mathbf{B}$ unique?

3. A matrix $\mathbf{A} \in \mathbf{C}^{n \times n}$ is simple if there exists a set of n linearly independent eigenvectors for $\mathbf{A}$.

   (a) Show that a matrix $\mathbf{A}$ is simple if and only if there is a non-singular matrix $\mathbf{X}$ such that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is diagonal. Show also that, in this situation, the columns of $\mathbf{X}$ are right eigenvectors for $\mathbf{A}$ and the rows of $\mathbf{X}^{-1}$ are left eigenvectors for $\mathbf{A}$ (i.e., eigenvectors for $\mathbf{A}^H$).

   (b) Let $\mathbf{A}$ be a simple matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that there are right eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ and left eigenvectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ such that

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \mathbf{y}_i^H.$$

4. Prove the Schur decomposition theorem: If $\mathbf{A} \in \mathbf{C}^{n \times n}$ then there exists a unitary matrix $\mathbf{Q}$ such that

$$\mathbf{Q}^H \mathbf{A} \mathbf{Q} = \mathbf{T},$$

where $\mathbf{T}$ is upper triangular. Use this theorem to prove that a Hermitian symmetric matrix $\mathbf{A}$ has a complete set of eigenvectors, and that the matrix of eigenvectors may be used to diagonalize $\mathbf{A}$.

5. For a matrix $\mathbf{A} = (a_{ij})$ in $\mathbf{C}^{n \times n}$ the trace of $\mathbf{A}$, $tr(\mathbf{A})$ is defined to be $tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$.

   (a) For $\mathbf{E}, \mathbf{D}$ in $\mathbf{C}^{n \times n}$ show that $tr(\mathbf{ED}) = tr(\mathbf{DE})$.

   (b) Show that if $\mathbf{A}$ and $\mathbf{B}$ are similar matrices then they have the same trace.

   (c) Show that the trace of a matrix equals the sum of its eigenvalues.

   (d) Show that the determinant of a matrix equals the product of its eigenvalues.

6. Let $\mathbf{A} \in \mathbf{R}^{m \times 2}$ be given, with $\mathbf{A}$ partitioned as

$$\mathbf{A} = [a \quad a + r], a, r \in \mathbf{R}^m, ||a||_2 = 1, ||r||_2 = \epsilon$$

   where ϵ is less than the machine epsilon. Explain why using a normal equations approach to solving a least squares problem with this coefficient matrix would be potentially disastrous.

7. Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be given, symmetric and positive definite. Define $\mathbf{A}_0 = \mathbf{A}$, and consider the sequence defined by

$$\mathbf{A}_k = \mathbf{G}_k \mathbf{G}_k^T$$
$$\mathbf{A}_{k+1} = \mathbf{G}_k^T \mathbf{G}_k$$

   where $\mathbf{G}_k$ is the Cholesky factor for $\mathbf{A}_k$. Prove that the $\mathbf{A}_k$ all have the same eigenvalues.

8. Define the *condition number*, κ, of a matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$. Prove that, for any singular $\mathbf{B} \in \mathbf{R}^{n \times n}$,

$$\kappa(\mathbf{A})^{-1} \geq ||\mathbf{A} - \mathbf{B}|| / ||\mathbf{A}||$$

   Explain what this result implies about the condition number of a matrix that is close to being singular. More generally, what does the condition number tell us about the stability of solutions to a linear system?

University of Alabama System
Joint Doctoral Program in Applied Mathematics
Joint Program Examination

Time: Three and One Half Hours

September, 1993

Linear Algebra and Numerical Linear Algebra

Time: Three and One Half Hours

Instructions: Do 7 of the 8 problems given. Include all the work for full credit. Be sure to indicate which 7 are to be graded.

(1). Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

(1.a) Find a matrix S such that $S^{-1}AS$ is diagonal.

(1.b) Show that, for any real number t,

$$e^{tA} = S e^{tD} S^{-1},$$

where $D = S^{-1}AS$ and

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!},$$

and hence compute the matrix e^{tA}.

(1.c) Show that the vector function $y(t) = e^{tA} y_0$ is a solution of the linear differential equation initial value problem

$$dy(t)/dt = A\,y(t), \quad y(0) = y_0,$$

and hence solve this initial value problem when $y_0 = (1, 2)^T$.

(2). Let V denote the vector space of all polynomials over R in two variables x and y of degree at most 2. Consider the mapping $T: V \to V$ defined by

$$T(p) = \frac{\partial}{\partial y} p \quad \forall\, p \in V.$$

Find both a Jordan canonical form and a Jordan canonical basis β for T.

(3). (3.a) Let $A_{nxn}B_{nxn} = 0$. Show: $\text{rank}(A) + \text{rank}(B) \le n$.

(3.b) Give two examples to verify that it is possible to have

$$\text{rank}(A) + \text{rank}(B) = n \quad \text{and} \quad \text{rank}(A) + \text{rank}(B) < n$$

under the same condition that $A_{nxn}B_{nxn} = 0$.

(4). Let $P_n[x] = \{ a_{n-1} x^{n-1} + \dots + a_1 x + a_0 : a_i \text{ real constants, } x \text{ real variable}\}, n > 1$. Let D be the differential operator on $P_n[x]$, $Df(x) = f'(x)$.

(4.a). Show that $P_n[x]$ is a vector space under ordinary addition and scalar multiplication.

(4.b). Find a specific polynomial

$$p(t) = a_k t^k + \dots + a_1 t + a_0 \ne 0$$

with the smallest k satisfying $p(D) = 0$ on $P_n[x]$. You need to justify your choice of k.

(4.c). Define an inner product on $P_n[x]$ as follows.

$$\langle f, g \rangle = \sum_{k=0}^{n-1} a_k b_k,$$

for any $f(x) = a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ and $g(x) = b_{n-1} x^{n-1} + \dots + b_1 x + b_0$ in $P_n[x]$.

Find D*, the adjoint operator of D, defined in terms of elements in $P_n[x]$.

(5). (5.a). Derive the algorithm to solve the system

$$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i \quad (a_1 = c_n = 0,\ d_i \neq 0,\ i = 1, \ldots, n)$$

using the LU factorization without pivoting.

(5.b). Explain briefly the terms "sensitivity" and "stability" in the context of finding the solution x of a linear system Ax = b. Solve the system

$$\begin{bmatrix} 0.001 & 1.00 \\ 1.00 & 2.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 3.00 \end{bmatrix},$$

using the LU decomposition method obtained in (5.a) and chopped floating point arithmetic using base 10 and a three digit mantissa to obtain a computed solution $x^{\#}$. Find a matrix E such that $(A + E)x^{\#} = (1.00, 3.00)^T$. Comment on the stability of Gaussian elimination without pivoting.

(6). Let A be a nonsingular matrix. Backward error analysis of Gaussian elimination with partial pivoting tells us that, if $x^{\#}$ is the computed solution of $Ax = b$, then $x^{\#}$ is the exact solution of a nearby system: specifically, there exists E satisfying $\|E\|_\infty \leq C u \|A\|_\infty$ for which $(A + E)x^{\#} = b$. Here C is a scalar constant, and u denotes the machine precision. One method of improving the accuracy of a computed solution is iterative refinement. We consider the function $f: R^n \to R^n$ given by $f(x) = x + z_x$, where z_x is the computed solution of $A z_x = b - Ax$, where the quantity $r = b - Ax$ is assumed to be computed exactly. Note that z_x satisfies $(A + F) z_x = r$ exactly for some F with $\|F\|_\infty \leq C u \|A\|_\infty$.

(6.a). Prove that if $\|A^{-1}F\|_\infty < 1$ then the matrix A + F is nonsingular and

$$\|(A+F)^{-1}\|_\infty \leq \frac{\|A^{-1}\|_\infty}{1 - \|A^{-1}\|_\infty \|F\|_\infty}.$$

(6.b). Let x* denote the exact solution of Ax = b. Show that

$$\|f(x) - f(x^*)\|_\infty \leq \frac{C\kappa_\infty(A)\,u}{1 - C\kappa_\infty(A)\,u}\|x - x^*\|_\infty,$$

where $\kappa_\infty(A)$ is the ∞-norm condition number of A.

(7). Describe in detail the eigenvalue computation algorithm known as " [explicit] QR with shift", as applied to a known upper Hessenburg matrix A_0. Your answer should include a description as to how the QR factorization is computed, how the matrix RQ is computed, and the strategies for choosing the shift. Compute by hand one [explicit] QR step with shift $\alpha = -2$ on the matrix

$$A_0 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

(8). (8.a) Let A be a real singular square matrix. Consider the linear system $A x = b$, where b is assumed to be in the range of A. Use the singular value decomposition method to construct a solution to $A x = b$ with the smallest length.

(8.b) If b is not in the range of A, then use the singular value decomposition method to construct a " least squares solution " for the linear system $A x = b$.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
TIME: THREE AND ONE HALF HOURS

May, 1993

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Instructions: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. (i) Let V be a vector space over R. Define the term "inner product on V". Prove the theorem of Pythagoras: For v_1 and v_2 in an inner product space V with v_1 orthogonal to v_2 we have

$$||v_1 + v_2||^2 = ||v_1||^2 + ||v_2||^2.$$

(ii) Let V be an inner product space, and let T be a linear operator on V. Define the term "adjoint operator T^*" Prove that

$$R(T^*)^{\perp} = N(T).$$

2. Let A be an $n \times n$ matrix with $A^2 = A$. Show that

$$\text{rank}\,(A) + \text{rank}\,(I_n - A) = n,$$

where I_n is the unit matrix.

3. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdot + a_1 t + a_0.$$

(i) Prove that A is invertible if and only if $a_0 \neq 0$.
(ii) Prove that if A is invertible then

$$A^{-1} = -\frac{1}{a_0}[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n].$$

(iii) Use part (ii) to compute A^{-1} for

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}.$$

4. Let $V = \text{span}(e^x, xe^x, x^2e^x, e^{3x})$. Define $T : V \to V$ by $T(f) = f''$ (the second derivative of f). Find both a Jordan canonical form and a Jordan canonical basis β for T.

5. (i) Define the term "least squares solution " for a linear system $Ax = b$, where $A \in \mathrm{R}^{m \times n}$. In some situations, a least squares solution for the system $Ax = b$ can be computed accurately, but not from the noraml equations, $A^T Ax = A^T b$. Explain, using the example

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}.$$

(ii) Assume in addition that $m \geq n$, rank $(A) = n$. Suppose an orthogonal matrix $Q \in \mathrm{R}^{m \times m}$ has been computed such that

$$Q^T A = R \equiv \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \begin{matrix} n \\ m-n \end{matrix}$$

is upper triangular. Derive the least squares equations to solve $Ax = b$ using the factorization above. Prove that the least squares solution always exists and is unique.
(iii) Use (ii) to solve the system $Ax = b$ with

$$A = \begin{bmatrix} -5 & -2 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}.$$

(Hint: you can choose $A^T = I - 2\mathrm{u}\mathrm{u}^T/\mathrm{u}^T\mathrm{u}$ for appropriate $\mathrm{u} \in \mathrm{R}^3$.)

6. Recall a perturbation theorem for the eigenvalues: If μ is an eigenvalue of the perturbed matrix $A + E \in \mathrm{C}^{nxn}$ and $X^{-1}AX = D \equiv$

$\text{diag}(\lambda_1, \cdots, \lambda_n)$ then

$$\min_i |\lambda_i - \mu| \le \kappa_2(X) \|E\|_2.$$

Explain why this theorem implies that the symmetric eigenvalue problem is less sensitive to rounding errors than the unsymmetric eigenvalue problem.

7. (i) Using base 10 ($\beta = 10$), 2 digit (t=2) chopping arithmetic, solve the system

$$\begin{bmatrix} 11 & 15 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

via LU factorization.
(ii) Do one step of iterative improvement using t = 4 arithmetic to compute the residual.
(iii) Give a heuristic explanation of iterative improvement as applied to a linear system Ax = b. In particular, compare the relative error produced by Gaussian elimination (with pivoting) and the relative error ultimately produced by iterated improvement. Under what circumstances (if any) will the iterative improvement fail? Why?

8. The matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

has the eigenpairs $(\lambda_1, x_1) = (2, [1,0,0]^t)$, $(\lambda_2, x_2) = (-1, [0,1,-1]^t)$, and $(\lambda_3, x_3) = (3, [0,1,1]^t)$. Suppose the power method is applied with the starting vector $q^{(0)} = [1,1,-1]^t/\sqrt{3}$.
(i) Determine whether this process gives convergence to an eigenpair of A, and if so, to which one. Assume exact arithmetic.

(ii) Answer the same question for the inverse iteration (with the same starting vector $q^{(0)}$, and with A replaced by A^{-1}).
(iii) Answer the question (i) if the iterations are performed on computer.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
September, 1992

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Time allowed: Three and one half hours

Instructions: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. Define $P_n(\mathbf{R})$ to be the vector space of polynomials of degree less than or equal to n, defined on the real line. Further define

$$P_n^0(\mathbf{R}) = \{p \in P_n(\mathbf{R}) : p(0) = 0\}.$$

Let $T : P_n^0(\mathbf{R}) \to P_n^0(\mathbf{R})$ be defined by

$$T(p)(x) = p(x) + \int_0^x \frac{1}{t} p(t)\, dt.$$

   (a) Prove that $P_n^0(\mathbf{R})$ is a subspace of $P_n(\mathbf{R})$.

   (b) Is T one-to-one? (Give a proof.)

   (c) Is T onto? (Give a proof.)

   (d) Is T an isomorphism? (Give a proof.)

2. Determine all possible Jordan canonical forms for a linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^5$ whose characteristic polynomial is

$$p(t) = (t-2)^3(t-5)^2.$$

3. For this problem, let V be an arbitrary inner product space, with a subspace $S \subset V$, $\dim S < \dim V$.

   (a) Let $x \in V$ be arbitrary, and define $p \in S$ to be the orthogonal projection of x into S. Show that

$$||x - p||_2 \leq ||x - q||_2, \qquad \forall q \in S.$$

   (b) Now let v_1, v_2, both in V, be orthogonal. Prove the Pythagorean Theorem:

$$||v_1 + v_2||^2 = ||v_1||^2 + ||v_2||^2.$$

4. Let $A \in \mathbf{R}^{m \times n}$ be given.

   (a) Define the rank of A;

   (b) Prove that $\text{rank}(A) \leq \min(m, n)$;

(c) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 2 & 8 & 1 & 5 \\ -1 & -4 & 3 & 1 \\ 0 & -3 & 4 & 4 \end{bmatrix}$$

What is a basis for the range (or column space) of A?

(d) Given $B \in \mathbf{R}^{n \times k}$, prove that

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)).$$

5. (a) Define $k_\infty(A)$, the condition number of a matrix (with respect to inversion, and with respect to the infinity norm).

(b) Assume that A and $A + E$ are nonsingular, with

$$\begin{aligned} Ax &= b \\ (A+E)x_c &= b \end{aligned}$$

Prove that

$$\frac{||x - x_c||_\infty}{||x_c||_\infty} \leq k_\infty(A) \frac{||E||_\infty}{||A||_\infty}.$$

(c) Let

$$A = \begin{bmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{bmatrix}, \quad E = \begin{bmatrix} 0.9 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}, \quad x = (1,0)^T$$

Use the estimate from (b) to find a lower bound for $k_\infty(A)$.

6. Assume that the matrix $A \in \mathbf{R}^{n \times n}$ has the following partitioned form:

$$A = \begin{bmatrix} B & b \\ a^T & \alpha \end{bmatrix}$$

where $B \in \mathbf{R}^{(n-1)\times(n-1)}$, $a, b \in \mathbf{R}^{n-1}$, and $\alpha \in \mathbf{R}$. Assume that a decomposition of B exists:

$$B = LU$$

where L is nonsingular and lower triangular, and U is non-singular and upper triangular.

(a) Show how to construct a similar decomposition for A.

(b) Under what conditions does this decomposition exist?

7. Assume the following: if $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, and $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ are the eigenvalues of the Hermitian matrices A, B, and $C = A + B$ (respectively), then

$$\alpha_i + \beta_1 \leq \gamma_i \leq \alpha_i + \beta_n, \qquad \forall i.$$

Use this to prove the following: If A, B in $\mathbf{C}^{n \times n}$ are Hermitian, with A positive definite and $||A^{-1}||_2 ||B||_2 < 1$, then $A + B$ is also positive definite. Hint: recall that for Hermitian A, $||A||_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

8. Let $A \in \mathbf{C}^{n \times n}$, with $v \in \mathbf{C}^n$ an approximate eigenvector of unit length, and $\lambda \in \mathbf{C}$ the corresponding approximate eigenvalue. Denote the residual by $r = Av - \lambda v$. Prove that there exists a matrix $E \in \mathbf{C}^{n \times n}$, with $||E||_2 \leq ||r||_2$, such that

$$(A + E)v = \lambda v,$$

i.e., (λ, v) is an exact eigenpair for the perturbed problem.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
May, 1992

Linear Algebra and Numerical Linear Algebra

Instructions: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. (i) Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_j\}$ be a basis for the subspace $S \subset \mathbb{R}^n$. Let the vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m \in S$ be linearly independent. Prove that there is an integer $k, 1 \leq k \leq j$, such that the vectors $\mathbf{a}_k, \mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_m$ are linearly independent.

   (ii) Use the result of (i) to prove that any basis for $\mathbb{R}^n$ must have exactly n elements.

2. (i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that there is a unique $m \times n$ matrix $\mathbf{A}$ such that $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (ii) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function defined by

$$f(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} 2x_2 + x_3 \\ x_1 - 4x_2 \\ 3x_1 \end{pmatrix}.$$

   Prove that f is a linear transformation and find the matrix representation of f with respect to the basis $B = \{\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$.

3. Let V be a vector space and T be a linear operator on V.

   (i) Prove that $T^2 = T_0$, where T_0 is the zero operator, if and only if $\mathcal{R}(T) \subseteq \mathcal{N}(T)$ where $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are the range and the null space of T respectively.

   (ii) Define the determinant of T, $\det(T)$, and prove that it is well-defined, that is, independent of a choice of basis.

4. (i) Define the term "symmetric and positive definite" (SPD) matrix.

   (ii) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Prove that $\mathbf{A}$ is an SPD matrix if and only if there is a unique lower triangular matrix $\mathbf{L}$ with positive main diagonal elements such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

   (iii) Determine whether or not the matrix

$$\mathbf{A} = \begin{pmatrix} 36 & 30 & 18 \\ 30 & 41 & 23 \\ 18 & 23 & 12 \end{pmatrix}$$

   is an SPD matrix. Give your reason.

5. (i) Let $\mathbf{A}_n = \begin{pmatrix} 1 & 2 \\ 2 & 4 + \frac{1}{n^2} \end{pmatrix}$. Find $\mathbf{A}_n^{-1}$ using Gaussian elimination.

   (ii) For the matrix $\mathbf{A}_n$ in (i) compute the condition number $\kappa_\infty(\mathbf{A}_n)$ in ∞-norm.

   (iii) Suppose that the systems $\mathbf{A}_n\mathbf{x} = \mathbf{b}$, $n = 1, 2, \cdots$, are being solved for some $\mathbf{b} \in \mathbb{R}^2$, where $\mathbf{A}_n$ is given in (i) and that a computer with base 2 and with 23 digits mantissa is being used. Chopping arithmetic is used in the computer. For what value(s) of n can the computed solution(s) not be trusted? Why?

6. (i) Define the term "least squares solution" for a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ real matrix with $m \geq n$. Prove that this least squares solution always exists and that it is unique if and only if the columns of $\mathbf{A}$ are linearly independent vectors.

(ii) Given data points $\{(t_i, y_i) : 1 \leq i \leq n\}$, the least squares line of best fit $y = a + bt$ for this data is obtained by using values for a and b that minimize the least squares error function

$$E(a, b) = \sum_{i=1}^{n} (y_i - (a + bt_i))^2.$$

Explain how the vector $\mathbf{x} = (a, b)^T$ may be obtained as a least squares solution of linear system.

7. (i) Define the term "Hermitian symmetric matrix".

(ii) Prove that the eigenvalues of a Hermitian symmetric matrix in $\mathbb{C}^{n \times n}$ are always real.

(iii) State the Schur decomposition theorem on the unitary triangularization of a matrix $\mathbf{A}$ in $\mathbb{C}^{n \times n}$. Use this theorem to prove that a Hermitian symmetric matrix $\mathbf{A}$ has a complete set of eigenvectors, and that the matrix of eigenvectors may be used to diagonalize $\mathbf{A}$.

8. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ have a complete set of eigenvectors and have eigenvalues λ_i, $1 \leq i \leq n$, satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Let $\mathbf{x}_1$ be the normalized eigenvector corresponding to λ_1, and let $\mathbf{q}^{(0)} \in \mathbb{C}^n$ be any vector satisfying $\mathbf{q}^{(0)H}\mathbf{x}_1 \neq 0$. Define, for $k \geq 1$,

$$\mathbf{z}^{(k)} = \mathbf{A}\mathbf{q}^{(k-1)}$$
$$\mathbf{q}^{(k)} = \frac{\mathbf{z}^{(k)}}{||\mathbf{z}^{(k)}||}$$
$$\lambda^{(k)} = \mathbf{q}^{(k)H}\mathbf{A}\mathbf{q}^{(k)}$$

(i) Prove that for $k \geq 1$,

$$\mathbf{q}^{(k)} = \mathbf{A}^k\mathbf{q}^{(0)}/||\mathbf{A}^k\mathbf{q}^{(0)}||.$$

(Hint: use induction on k.)

(ii) Let $\mathbf{q}^{(k)} = \alpha_k\mathbf{x}_1 + \mathbf{p}_k$ where $\mathbf{p}_k \in \{\mathbf{x}_1\}^{\perp}$. Show that $|\alpha_k|$ approaches one and $||\mathbf{p}_k||$ approaches zero, as k approaches infinity.

(ii) Show that

$$\lim_{k \to \infty} \lambda^{(k)} = \lambda_1.$$

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
October, 1991

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBR

Instructions: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. For a subspace $S \subset \mathbf{R}^n$ the orthogonal of S is defined by

$$S^{\perp} = \{y \in \mathbf{R}^n : y^T x = 0 \forall x \in S\}.$$

(i) Prove that, if $S \subset \mathbf{R}^n$ is a subspace, then $S^{\perp}$ is also a subspace and show that $\dim(S) + \dim(S^{\perp}) = n$.

(ii) Let $R(A)$ and $N(A)$ denote the range space and null space of a matrix A, respectively. If $A \in \mathbf{R}^{m \times n}$ show that $R(A)^{\perp} = N(A^T)$.

2. Determine all possible Jordan canonical forms for a linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^5$ whose characteristic polynomial is

$$p(\lambda) = (\lambda - 2)^3(\lambda - 5)^2.$$

3. Let V be an n-dimensional vector space. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ be two bases for V.

(i) Let $f : V \to V$ be an invertible linear transformation. With a given choice of basis for V let A be the matrix representing f. Show that A is invertible and that f^{-1} is a linear transformation represented by A^{-1}.

(ii) Two $n \times n$ matrices B, C are similar if there is an $n \times n$ invertible matrix A such that $B = ACA^{-1}$. Suppose that $f : V \to V$ is a linear transforation represented by the matrix B with respect to $\{v_1, v_2, \ldots, v_n\}$ and by the matrix C with respect to $\{w_1, w_2, \ldots, w_n\}$. Show that B and C are similar.

(iii) Show that there is a linear transformation $f : V \to V$ such that if $x \in V$ and $s = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i w_i$, then $f(a) = b$, where

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

4. Consider a linear system $Ax = b$, where A is a real non-singular matrix of order n.

   (i) Discuss the algorithm known as Gaussian elimination with partial pivoting for solving the linear system. Include in your answer an indication as to how you would compute the inverse matrix A^{-1}.

   (ii) If A is a strictly diagonally dominant matrix show that Gaussian elimination with no pivoting can be applied to solve the linear system and a zero pivot will not be encountered.

5. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a square symmetric matrix A. Let x_1 be the eignevector corresponding to λ_1 with the property $||x_1||_2 = 1$. Define the matrix $B = A - \lambda_1 x_1 x_1^T$. Prove that B has eigenvalues $0, \lambda_2, \cdots, \lambda_n$.

6. Define the condition number, $k_\infty(A)$, of a matrix A of order n, with respect to the infinity norm on $\mathbf{R}^n$.

   (i) Consider the linear system $Ax = b$. Let x_c be a computed solution to this system and define the residual $r = b - Ax_c$. Prove that

$$\frac{||x - x_c||_\infty}{||x||_\infty} \leq k_\infty(A) \frac{||r||_\infty}{||b||_\infty}.$$

   (ii) If $||b||_\infty = 1$ and $k_\infty(A) = 10^4$, and the system $Ax = b$ is solved in single precision (8-digit) arithmetic, what can you conclude about the relative error in x? (Hint: you may assume that the matrix A is entered exactly into a machine and try to estimate the absolute error, $||r||_\infty$, in b.)

   (iii) Let $D = \text{diag}(10^{-1}, \ldots, 10^{-1})$ be a diagonal matrix of order n. Compute $\det(D)$, the determinant of D, and $k_\infty(D)$ as a function of n. In light of the fact that the determinant of a matrix is zero if and only if the matrix is singular, comment on the suggestion that the smallness of the determinant be used as a measure of ill-conditioning for the matrix.

7. Let $\lambda(A)$ be the set of all eigenvalues of A. If a matrix A can be partitioned as

$$A = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

show that $\lambda(A) = \lambda(T_{11}) \cup \lambda(T_{22})$.

8. Let A be a complex $n \times n$ matrix, and v be a complex n-vector. Prove that, if $r = Av - \mu v$, then $||r||_2$ is minimized when $\mu = \frac{v^H A v}{v^H v}$, where v^H denotes the conjugate transpose of v.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
May, 1991

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

<u>Instructions</u>: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+b \\ a-3b \end{bmatrix}$$

(i) Show that T is linear.

(ii) Find a basis for $N(T)$, the null space of T, and a basis for $R(T)$, the range of T. Is T one-to-one? Is T onto?

(iii) Determine $[T]_\beta$, the matrix of T with respect to the basis, β, where

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

and $[T]_{\beta'}$ where

$$\beta' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

(iv) Show that $[T]_\beta$ is similar to $[T]_{\beta'}$ by finding a matrix Q such that $Q^{-1}[T]_\beta Q = [T]_{\beta'}$.

2. Let A be a complex-valued $n \times n$ matrix.

(i) What do we mean by "A is Hermitian"?

(ii) Show that if A is Hermitian, then all the eigenvalues of A are real, and eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(iii) Show that no generalized eigenvector of A has rank greater than 1. Recall that x is a generalized eigenvector of A with rank m corresponding to eigenvalue λ if $(A-\lambda I)^m x = 0$ but $(A-\lambda I)^{m-1}x \neq 0$.

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a matrix J in Jordan canonical form and a matrix Q such that $J = Q^{-1}AQ$.

4. Let A be an $n \times n$ matrix whose characteristic polynomial is $(a - \lambda)^n$.

(i) State the Cayley-Hamilton theorem and use it to show that, if $P(\lambda)$ is a polynomial, then

$$P(A) = \sum_{k=0}^{n-1} \frac{P^{(k)}(a)}{k!}(A - aI)^k.$$

(ii) Use (i) to compute $A^{24} - 3A^{15}$ where

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix}.$$

5. Let $A \in \mathbf{R}^{m \times n}$, $x \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$.

(i) Define "least squares solution" for the linear system $Ax = b$.

(ii) Derive the normal equations for solving $Ax = b$.

(iii) Use (ii) to find the "least squares solution" of

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix}.$$

6. Define the condition number, $K_\infty(A)$, of a matrix A with respect to inversion, and with respect to the infinity norm.

(i) Let the matrices A and $A + \Delta A$ be nonsingular. If, for a given vector b,

$$Ax = b$$
$$(A + \Delta A)(x + \Delta x) = b$$

prove that

$$\frac{||\Delta x||_\infty}{||x + \Delta x||_\infty} \leq K_\infty(A)\frac{||\Delta A||_\infty}{||A||_\infty}$$

(You may assume that $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$.)

(ii) Let

$$A = \begin{bmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0.9 & 0.2 \\ 0.3 & 0.4 \end{bmatrix},$$

and $x = (1,0)^T$. Use the estimate in (i) above to find a lower bound for $K_\infty(A)$.

7. Let $u \in \mathbf{R}^n$ be a given vector and

$$P = I - \frac{2}{u^T u} u u^T$$

be a Householder reflector matrix.

(i) Prove that P is orthogonal.

(ii) Let x be given and let $x = v + w$ where v lies along the vector u, and w is orthogonal to u. Show that $Px = -v + w$, and explain why P is called a "reflector" matrix.

(iii) For a given matrix A, explain briefly how to use Householder matrices to compute the decomposition $A = QR$ where Q is orthogonal and R is upper triangular.

8. Describe in detail the eigenvalue computation algorithm known as "explicit QR with shift", as applied to a known upper Hessenburg matrix A_0. Your answer should inculde a description as to how the QR factorization is computed, how the matrix RQ is computed, and the strategies for choosing the shift. Compute by hand one explicit QR step with shift $\alpha = -2$ on the matrix

$$A_0 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
October, 1990

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Instructions: Do any 7 of the 8 problems given. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. Let A and B be linear transformations on an n-dimensional vector space. Prove that

   (i) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

   (ii) If A is invertible, then $\text{rank}(AB) = \text{rank}(BA) = \text{rank}A$.

   (iii) If $AB = 0$, then $\text{rank}(A) + \text{rank}(B) \leq n$.

2. Let V be the vector space of all polynomials of degree ≤ 2 with real coefficients and in a variable x.

   (i) Show that $\{f_0, f_1, f_2\} \subset V$ where $f_r(x) = x^r$, for $0 \leq r \leq 2$, is a basis for V.

   (ii) Let $T : V \longrightarrow V$ be defined by

$$\forall f \in V, \quad T(f)(x) = f(x+1).$$

   (a) What is $T^n(f)(x)$?

   (b) Write down the matrix of T relative to the basis $\{f_0, f_1, f_2\}$.

   (c) Show that $(T - I)^3 = 0$ where I is the 3×3 identity matrix.

   (d) Deduce from (c) that $\forall f \in V$

$$f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0.$$

3. Let $A \in \mathbf{C}^{n \times n}$ (complex-valued $n \times n$ matrix) and A^H be the conjugate transpose of A.

   (i) What do we mean by "A is normal"?

   (ii) Show that if A is normal, then $A - \lambda I$ is normal for any $\lambda \in \mathbf{C}$, and $||Ax|| = ||A^H x||$ for all x.

   (iii) Use (ii) to show that if A is normal, then A and A^H have same eigenvectors and if x and y are eigenvectors of A belonging to distinct eigenvalues, then they are orthogonal.

4. (i) Let $A \in \mathbf{C}^{n \times n}$, and $\lambda_i \in \mathbf{C}$, $i = 1, 2, \ldots, n$ be the eigenvalues of A. Show that $\det(A) = \Pi_{i=1}^n \lambda_i$ and $\text{tr}(A) = \sum_{i=1}^n \lambda_i$.

   (ii) Show that if $A, B \in \mathbf{C}^{n \times n}$ are similar matrices, then they have the same characteristic polynomial. Deduce that $\det(A) = \det(B)$, and $\text{tr}(A) = \text{tr}(B)$.

5. Let

$$A = \begin{bmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{bmatrix}, b = \begin{bmatrix} 4.1 \\ 9.7 \end{bmatrix}, \text{ and } b' = \begin{bmatrix} 4.11 \\ 9.70 \end{bmatrix}$$

(i) Find a permutation matrix P, a unit lower triangular matrix L and an upper triangular matrix U such that

$$PA = \begin{bmatrix} 9.7 & 6.6 \\ 4.1 & 2.8 \end{bmatrix} = LU \qquad (*)$$

and use (*) to solve $Ax' = b'$. (It's clear that $A\binom{1}{0} = b$).

(ii) Find a lower bound of $K_\infty(A)$, the condition number of A with respect to $||\cdot||_\infty$ norm, by using the result in (i) and an appropriate theorem to be stated.

6. (i) State the theorem, due to Shur, that ensures the diagonalization of the symmetric matrix $A \in \mathbf{R}^{m \times n}$.

(ii) Find the Shur decomposition of

$$A = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix}.$$

7. (i) Define "least squares solution" for a linear system $Ax = b$.

(ii) Assume that $A \in \mathbf{R}^{m \times n}$, $m \geq n$ and $\text{rank}(A) = n$. Consider the factorization of A into Q_1R_1 where Q_1 has orthonormal columns and R_1 is upper triangular, of order n, and non-singular, or into QR where Q is orthogonal and

$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

is an $m \times n$ matrix. Use either factorization of A to derive the least squares equations for $Ax = b$ in terms of R_1.

(iii) Use (ii) to find a least square solution for $Ax = b$ with

$$A = QR = \begin{bmatrix} .6 & .8 & 0 \\ .8 & -.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix},$$

or

$$A = Q_1 R_1 = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

8. Let $A \in \mathbf{C}^{m \times n}$ and

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^H$$

be a singular value decompostion (SVD) of the complex matrix A where $U = [u_1, u_2, \ldots, u_m] \in \mathbf{C}^{m \times m}$, $V = [v_1, v_2, \ldots, v_n] \in \mathbf{C}^{n \times n}$ are unitary, and

$$\sum = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$$

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$$

and

$$0 \leq p \leq \min(m, n).$$

Prove that

(i) $Av_i = \sigma_i u_i$ for $1 \leq i \leq p$;

(ii) $\text{null}(A) = \text{span}\{v_{p+1}, \ldots, v_n\}$;

(iii) $\text{range}(A) = \text{span}\{u_1, u_2, \ldots, u_p\}$;

(iv) $||A||_2 = \sigma_1$.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
May, 1990

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Instructions: The examiners will grade 7 problems. Do at least 3 problems of 1–5 and at least 3 of 6–10 Include all work for full credit. If you work more than 7 problems, indicate the problems you want graded.

1. Let V and W be vector spaces over F and let $T : V \longrightarrow W$ be an invertible linear transformation. Prove that

   (a) $N(T) = \{0\}$, where $N(T)$ is the null space of T, and

   (b) T preserves linear independence; i.e. if S is a linearly independent subset of V then $T(S)$ is a linearly independent subset of W.

2. Prove that if A and B are real $m \times n$ matrices then

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

3. Prove the Cauchy-Schwarz inequality for a complex inner product space V:

$$|\langle x, y \rangle| \leq ||x|| \cdot ||y||$$

   for all x, $y \in V$.

4. Recall that the generalized eigenspace of a linear operator $T : V \longrightarrow V$ corresponding to an eigenvalue λ of T is

$$K_\lambda = \{x \in V : (T - \lambda I)^p(x) = 0 \text{ for some positive integer } p\}$$

   Prove that

   (a) K_λ is a subspace of V and

   (b) K_λ is T-invariant; i.e. $T(K_\lambda) \subset K_\lambda$.

5. Let $a, b \in \mathbf{R}$ and $A = aI + bJ$ where I is the $n \times n$ identity matrix and J is the $n \times n$ all ones matrix. Find the characteristic polynomial, eigenvalues, and eigenspaces of A.

6. (a) Describe in detail the Gauss elimination procedure, with partial pivoting, for solving a linear system $Ax = b$, where A is an $n \times n$ non-singular matrix and x and b are vectors in $\mathbf{R}^n$. Include in your answer how one obtains the decomposition $PA = LU$, where L is a unit lower triangular matrix, U is upper triangular, and P is an appropriate permutation matrix. Describe also how one computes x, given b, L and U.

(b) Describe the method of iterative refinement, used to improve the accuracy of the solution x obtained in (a) above.

7. Define the condition number, $K(A)$, of a matrix A (with respect to inversion, and with respect to a given norm, $||\cdot||$).

   (a) Calculate $K_\infty(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

   (b) Let A be a non-singular matrix. Show that $K_2(A) = 1$ if and only if A is a non-zero scalar multiple of an orthogonal matrix.

   (c) Let $D = \text{diag}(10^{-1}, \ldots, 10^{-1})$ be a diagonal matrix of order n. Computer $\det(D)$, and $K_2(D)$ as a function of n. Comment on the suggestion that the determinant be used as a measure of condition.

   (d) Let x be the solution of the system $Ax = b$, where A is a non-singular $n \times n$ matrix. Let $y = x + \Delta x$ be the solution to $Ay = b + \Delta b$. Prove that, if b and x are non-zero,

$$\frac{||\Delta x||}{||x||} \leq K(A)\frac{||\Delta b||}{||b||}.$$

8. Define the term "least squares solution" for a linear system $Ax = b$. Prove that this least squares solution always exists and that it is unique if and only if the null space of the matrix A contains only the zero vector. Show also that the least squares solution may be found by solving the normal system $A^T Ax = A^T b$.

9. Prove the Gerschgorin theorem: If $\lambda(A)$ denotes the set of eigenvalues of a matrix $A = (a_{ij})$, then

$$\lambda(A) \subset \bigcup_{i=1}^{n} D_i$$

where

$$D_i = \{z \in C : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

Use this theorem to show that a real symmetric, strictly diagonally dominant matrix with positive diagonal elements is positive definite.

10. Prove the singular value decomposition theorem: Let A be an $m \times n$ matrix. Then there exist unitary matrices U and V such that

$$V^H AU = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

where $\sum = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Let A be a square matrix with singular values $\{\sigma_i : 1 \leq i \leq n\}$. Prove that

$$||A||_2 = \max_{1 \leq i \leq n} \sigma_i.$$

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM
IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
September 1989

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Instructions: The examiners will grade at most 7 problems. Include all work for full credit. Be sure to indicate which 7 are to be graded.

1. Let $f(x) = a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$ be the characteristic polynomial of a real $n \times n$ matrix A.

   (a) Show that if $a_0 \neq 0$ then A is invertible and A^{-1} can be written as a polynomial in A.

   (b) Use part (a) to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}.$$

2. Let $\mathrm{tr}(A)$ denote the trace of a square matrix A.

   (a) Prove that if A and B are $n \times n$ matrices then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

   (b) Use part (a) to show that similar matrices have the same trace.

3. Let

$$A = \begin{bmatrix} 3 & 0 & 2 \\ -3 & 1 & -2 \\ -1 & 0 & 0 \end{bmatrix}$$

   Find a matrix J in Jordan canonical form and a matrix Q such that $Q^{-1}AQ = J$.

4. Let V be a real inner product space and let W be a subspace of V having an orthonormal basis $\{w_1, w_2, \ldots, w_n\}$.

   (a) Show that if x is in W then $x = \sum_{i=1}^{n}(x, w_i)w_i$.

   (b) Show that if x is in V and $y = \sum_{i=1}^{N}(x, w_i)w_i$, then $||x-y|| \leq ||x-z||$ for all z in W.

5. A matrix $A = (a_{ij})$ in $\mathbf{R}^{n\times n}$ is said to be column diagonally dominant if

$$|a_{jj}| > \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}|$$

for $j = 1, \ldots, n$. Show that if such a matrix is the coefficient matrix of a system of linear equations to be solved by Gaussian elimination, then no partial pivoting will be required. Hint: show that at each stage of the elimination process the square matrix in the lower right hand corner is still column diagonally dominant, ensuring that the pivotal element is already on the diagonal.

6. (a) Show that the eigenvalues of a real symmetric matrix are real.

   (b) Let A be a real symmetric $n \times n$ matrix with eigenvalues denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let x_1 be an eigenvector corresponding to λ_1 such that $(x_1)^T x_1 = 1$. Define

$$B = A - \lambda_1 x_1 (x_1)^T.$$

   Show that B has eigenvalues $0, \lambda_2, \cdots, \lambda_n$.

7. Define the 1-norm of a matrix A in $\mathbf{R}^{m\times n}$ by

$$||A||_1 = \sup_{x\in\mathbf{R}^n} ||Ax||_1/||x||_1$$

where

$$||(x_1, x_2, \cdots, x_n)||_1 = |x_1| + |x_2| + \cdots + |x_n|.$$

Show that $||A||_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$.

8. (a) State the Singular Value Decomposition Theorem.

   (b) Define the condition number, $K(A)$, of a matrix A in $\mathbf{R}^{m\times n}$ with respect to a norm $||\cdot||$.

   (c) Prove that the condition number of an $n \times n$ nonsingular matrix A with respect to the 2-norm $||\cdot||_2$ is $K(A) = \sigma_1/\sigma_n$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ are the n singular values of A.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM
IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
May 1989

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

Instructions: Do any 7 of the 8 problems given. Do not turn in work on all 8 problems. Include all work for full credit.

1. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation.

   (a) Define the terms "null space" and "rank" of a linear transformation.

   (b) Prove that

   $$\dim(V) = \dim(N(T)) + \operatorname{rank}(T),$$

   where $N(T)$ denotes the null space of T.

2. Let α and β be distinct eigenvalues of a linear operator $T : V \to V$, where V is a vector space. Let A and B be linearly independent sets of eigenvectors corresponding to α and β, respectively. Prove that $A \cup B$ is linearly independent.

3. (a) Prove that similar matrices have the same characteristic polynomial.

   (b) List one matrix in Jordan canonical form from each similarity class of matrices having characteristic polynomial

   $$f(t) = (2 - t)^4(3 - t).$$

   Give the minimal polynomial for each of these matrices.

4. (a) Let $V = C[0, 1]$ be the vector space of continuous real-valued functions defined on the interval $[0, 1]$, with inner product

   $$(f, g) = \int_0^1 f(t)g(t)\,dt.$$

   Find an orthogonal basis for the space spanned by $\{1, t^2, t^4\}$.

   (b) Prove that if W is a subspace of a finite-dimensional inner product space V, then

   $$(W^\perp)^\perp = W.$$

5. Let A be a real square matrix.

   (a) Prove that A is nonsingular if and only if 0 is not an eigenvalue of A.

(b) Define the term "diagonally dominant matrix", and prove that if A is a diagonally dominant matrix than A is nonsingular.

6. Let A be a real symmetric matrix, with eigenvalues λ_i, $1 \leq i \leq n$, satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

If x_1 is the eigenvector corresponding to λ_1, and z_0 is a vector satisfying $z_0^T x_1 \neq 0$, prove that

$$\lim_{k\to\infty} \frac{z_0^T A^k z_0}{z_0^T A^{k-1} z_0} = \lambda_1$$

7. For a square real matrix A, define $||A||_2$ as

$$||A||_2 = \max_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

Prove the following:

(a) If Q is orthogonal, then $||Qx||_2 = ||x||_2$ for all x.

(b) If Q is orthogonal, $||QA||_2 = ||A||_2$ for any A.

(c) $||A||_2 = \sqrt{\rho(A^T A)}$, where $\rho(B)$ denotes the spectral radius of B.

8. (a) Define the term "unitary matrix".

(b) If v^H denotes the conjugate transpose of the vector v, prove that $U = I - vv^H$ is unitary if and only if $||v||_2 = 2$.

(c) Let x and y be vectors. Prove that if $||x||_2 = ||y||_2$ and if the inner product (x, y) is real, then there exists a unitary matrix U of the form $I - vv^H$ such that $Ux = y$, for some vector v.

(d) Let

$$A = \begin{pmatrix} 4 & 4 & 1 \\ 3 & -2 & 7 \\ 0 & 3 & 1 \end{pmatrix}$$

Exploit (b) and (c) to find the factorization of the matrix A, such that $A = QR$, where Q is unitary and R is an upper triangular matrix.

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
January 1989

LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

<u>Instructions</u>: Do any 7 of the 8 problems given. Include all work for full credit.

1. (i) State and prove the Cayley-Hamilton Theorem.

   (ii) Suppose the $n \times n$ real matrix A satisfies a real polynomial equation $g(x) = 0$, where g has degree $m \leq n$. Suppose further that A satisfies no real non-zero polynomial equation of degree less than m and that $x - \alpha$ is a real factor of $g(x)$. Prove that α is an eigenvalue of A.

2. (i) Write down the symmetric matrix associated with the real quadratic form

$$ax^2 + 2bxy + cy^2$$

   (ii) By considering $\det(A - xI)$, or otherwise, show that the quantities $a + c$ and $b^2 - ac$ are invariant under orthogonal transformation.

   (iii) Given that:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   find θ in terms of a, b, and c, such that $T^{-1}AT$ is diagonal.

3. Find the Jordan canonical form for:

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

4. Define the terms *linear transformation* and *direct sum of vector spaces.*

   Let i be a fixed integer with $1 \leq i \leq n$, and let $P : \mathbf{R}^n \to \mathbf{R}^n$ be defined by projecting each vector onto the i-th coordinate axis. Prove that P is a linear transformation and find $\operatorname{Ker} P$, $\operatorname{Im} P$, and M, the matrix of P. Prove that $M(I - M) = 0$.

   Let $T : V \to V$ be any linear transformation such that $T^2 = T$. Show that $V = \operatorname{Ker} T \oplus \operatorname{Im} T$. Here the symbol "$\oplus$" represents direct sum.

5. Let $A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^n$ be given, and consider the linear system

$$Ax = b.$$

   Under what conditions on A and/or b does this problem have a solution, and when is it unique? Be as complete and precise as possible.

6. (i) Define the condition number, $\kappa(A)$, of a matrix A, with respect to a given norm $\|\cdot\|$. Prove that $\kappa(A) \geq 1$. Are any additional hypotheses needed for this last conclusion?

(ii) Calculate $\kappa_1(A)$, $\kappa_2(A)$, and $\kappa_\infty(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(iii) Consider the two systems below. Solve both, using 3-digit decimal arithmetic, and comment on your results.

$$\begin{bmatrix} 2.00 & 1.42 \\ 1.00 & 0.711 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8.00 \\ 4.00 \end{bmatrix}$$

$$\begin{bmatrix} 2.00 & 1.42 \\ 1.00 & 0.711 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 8.00 \\ 4.10 \end{bmatrix}$$

7. Let A be a given $n \times n$ nonsingular matrix, and assume a splitting of the form $A = M - N$, where M is non-singular. Let x be the solution of the problem $Ax = b$. Consider the iteration

$$Mx^{(k+1)} = b + Nx^{(k)}.$$

Show that the errors $e^{(k+1)} = x - x^{(k)}$ satisfy a relation of the form:

$$e^{(k+1)} = Ge^{(k)},$$

and that the residuals $r^{(k)} = b - Ax^{(k)}$ satisfy a relation of the form:

$$r^{(k+1)} = Hr^{(k)}$$

for appropriate matrices G and H. How are G and H related? Prove that $\rho(H) < 1$ if and only if $\rho(G) < 1$. (Here $\rho(A)$ is the spectral radius of the matrix A.)

8. Let $A \in \mathbf{R}^{n\times n}$ be given, with $A = A^T$, and assume the eigenvalues of A satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Let $z \in \mathbf{R}^n$ be given, with $z^T x_1 \neq 0$, where x_1 is one eigenvector corresponding to λ_1. Prove that

$$\lim_{k \to \infty} \frac{A^k z}{\lambda_1^k} = C x_1$$

Use this result to devise a reliable algorithm for computing (λ_1, x_1).

UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
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LINEAR ALGEGRA AND NUMERICAL LINEAR ALGEBRA

<u>Instructions</u>: Do any 6 of the 7 problems given. Each problem has equal weight, and each sub-part of each problem has equal weight. Include all work for full credit.

1. (i) Define the condition number, $\kappa(A)$, of a matrix A, with respect to a given norm $||\cdot||$. Prove that $\kappa(A) \geq 1$. Are any additional hypotheses needed for this last conclusion?

   (ii) Calculate $\kappa_1(A)$, $\kappa_2(A)$, and $\kappa_\infty(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   (iii) Let $D = \text{diag}(0.1, 0.1, \ldots, 0.1)$ be an $n \times n$ diagonal matrix. Compute $\det D$ and $\kappa_2(D)$ as a function of n. Comment on the possible use of the determinant as a measure of the condition of a linear system.

2. (i) Define the *spectral radius*, $\rho(A)$, of an $n \times n$ matrix A.

   (ii) Let A be a given $n \times n$ real matrix. Prove that

$$||A||_2 = \rho(A^T A)^{1/2}$$

   (iii) Let $||\cdot||$ be any operator norm, i.e., $||Ax|| \leq ||A||\,||x||$ for all matrices A and vectors x; show that $\rho(A) \leq ||A||$.

   (iv) Let Q be an orthogonal matrix. Show that $||Q||_2 = 1$, and that $||Qx||_2 = ||x||_2$ for any vector x.

3. Let $A \in \mathbf{R}^{nxn}$ with λ an eigenvalue for A, with corresponding eigenvector x. Let $y \approx x$. Show that the Rayleigh quotient for y yields an accurate approximation for λ, and that the assumption that A is symmetric makes this approximation more accurate.

4. Let A be symmetric and positive definite. Show that there exists a unique lower triangular matrix G such that $GG^T = A$.

5. Let A be a given $n \times n$ matrix, and assume a splitting of the form $A = M - N$, where M is non-singular. Let x be the solution of the problem $Ax = b$. Consider the iteration

$$Mx^{(k+1)} = b + Nx^{(k)}$$

Show that the errors $e^{(k+1)} = x - x^{(k)}$ satisfy a recurrence of the form:

$$e^{(k+1)} = Ge^{(k)}$$

and that the residuals $r^{(k)} = b - Ax^{(k)}$ satisfy a recurrence of the form:

$$r^{(k+1)} = Hr^{(k)}$$

for appropriate matrices G and H. How are G and H related? Prove that $\rho(H) < 1$ if and only if $\rho(G) < 1$. Can we use (5) to justify terminating the iteration when the residuals get small? Why/why not?

6. Consider the two systems below. Solve both, using 3-digit decimal arithmetic. Comment on your results.

$$\begin{bmatrix} 2.00 & 1.42 \\ 1.00 & 0.711 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8.00 \\ 4.00 \end{bmatrix}$$

$$\begin{bmatrix} 2.00 & 1.42 \\ 1.00 & 0.711 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 8.00 \\ 4.10 \end{bmatrix}$$

7. Let $A \in \mathbf{R}^{n \times n}$ be given, with $A = A^T$, and assume the eigenvalues of A satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Let $z \in \mathbf{R}^n$ be given, with $z^T x_1 \neq 0$, where x_1 is the eigenvector corresponding to λ_1. Prove that

$$\lim_{k \to \infty} \frac{A^k z}{\lambda_1^k} = Cx_1$$

Use this result to devise a reliable algorithm for computing (λ_1, x_1). Be sure to comment on the necessity (or lack thereof) of each assumption. Can any of them be relaxed? What happens if $|\lambda_1| = |\lambda_2|$ is allowed? Can the algorithm still be used effectively?