University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2019

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Let $\mathbf{A}$ be an $m \times m$ matrix, and let a_j be its j-th column. Prove the following inequality:
$$|\det\mathbf{A}| \leq \Pi_{j=1}^{m}\|a_j\|_2.$$

2. (a) Let P_2 be the vector space of all polynomials with complex coefficients of degree at most 2. Define the linear transformation $T : P_2 \rightarrow P_2$ by the rule $Tp_2(z) = p_2(z+h)$, $\forall z \in \mathbb{C}$, $h \in \mathbb{C}$, fixed. Find the matrix of T with respect to the monomial basis in P_2.
   (b) What can you say about the matrix of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $rank(T) = r$, if in the basis $x_1, ..., x_n$ of $\mathbb{R}^n$, $x_{r+1}, ..., x_n \in N(T)$, where $N(T)$ is the nullspace of T?

3. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. The corresponding dual norm $\|\cdot\|'$ is defined by formula $\|x\|' = \sup_{\|y\|=1} |y^*x|$. Prove that the $\|\cdot\|_{\ell_1}$ and $\|\cdot\|_{\ell_\infty}$ are dual to each other. Prove that $\|\cdot\|$ coincides with $\|\cdot\|'$ if $\|\cdot\|$ is the 2-norm.

4. Consider solving the linear system $Ax = b$, where A is an $m \times n$ matrix with $m \leq n$ (underdetermined case), by minimizing $\|x\|_{\ell_2}$ subject to $Ax = b$.

   (a) Show that if $A \in \mathbb{R}^{m\times n}$ is full (row) rank, where $m \leq n$, then AA^T is invertible. Then show that $x^* = A^T(AA^T)^{-1}b$ is a solution to $Ax = b$.

   (b) Along with part (a) and the solution $x^* = A^T(AA^T)^{-1}b$, show that $\|x\|_{\ell_2} \geq \|x^*\|_{\ell_2}$ and thus, x^* is the optimal solution to the minimization problem.

5. Let $x, y \in \mathbb{R}^n$ such that $x \neq y$ but $\|x\|_{\ell_2} = \|y\|_{\ell_2}$, show that there exists a reflector Q of the form $Q = I - 2uu^T$ where I is the identity matrix, $u \in \mathbb{R}^n$ and $\|u\|_{\ell_2} = 1$ such that $Qx = y$.

   (a) Let $A^T = [3\ \sqrt{11}\ 4]$ and $b^T = [2\ 4\ 6]$. Compute a reduced QR decomposition of A using Householder reflections, then solve the least square problem of $\min_x \|b - Ax\|_{\ell_2}$ and calculate the residual error $\|b - Ax\|_{\ell_2}$.
   (b) Write a pseudo code for QR factorization via Householder reflection matrices.

6. Suppose $\mathbf{A} \in \mathbb{R}^{n\times m}$ has full rank, that is, $rank(\mathbf{A}) = r = min(m, n)$. Let $\sigma_1 \geq ... \geq \sigma_r$ be the singular values of $\mathbf{A}$. Let $\mathbf{B} \in \mathbb{R}^{n\times m}$ satisfy $\|\mathbf{A} - \mathbf{B}\|_2 < \sigma_r$. Then $\mathbf{B}$ also has full rank. Suppose $\mathbf{A} \in \mathbb{R}^{n\times m}$ has full rank, that is, $rank(\mathbf{A}) = r = min(m, n)$. Let $\sigma_1 \geq ... \geq \sigma_r$ be the singular values of $\mathbf{A}$. Let $\mathbf{B} \in \mathbb{R}^{n\times m}$ satisfy $\|\mathbf{A} - \mathbf{B}\|_2 < \sigma_r$. Then $\mathbf{B}$ also has full rank.

7. Suppose that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. In (b) and (c) below you can use the Schur decomposition but must prove anything else you want to use.

   (a) State (without proof) the Schur decomposition of A.
   (b) Show the inequality $\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2$.
   (c) Show that if A is normal (i.e., $A^*A = AA^*$), then $\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$.
   (d) Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, symmetric. Determine $\|A\|_2$.

8. Let $A, Q_0 \in \mathbb{R}^{m\times m}$. Define sequences of matrices Z_k, Q_k and R_k by
$$Z_k = AQ_{k-1}, \quad Q_kR_k = Z_k, \qquad k = 1, 2, \cdots,$$
where Q_kR_k is an QR factorization of Z_k. Suppose $\lim_{k\rightarrow\infty} R_k = R_\infty$ exists.

   (a) Does it necessarily $\lim_{k\rightarrow\infty} Q_k = Q_\infty$ exist? Justify your answer.
   (b) Determine the eigenvalues of A in terms of R_∞ if $\lim_{k\rightarrow\infty} Q_k = Q_\infty$ exists.