University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 2013

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ defined by

$$T(a,b,c,d,e) = (2a, 2b, 2c+d, a+2d, b+2e)\,.$$

   (a) Find the characteristic and minimal polynomial of T.

   (b) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.

   (c) Find the Jordan form of T with respect to your basis.

2. Let U and V be subspaces of the finite dimensional inner product space $\mathbf{V}$.

   (a) Prove that $U^\perp \cap W^\perp = (U+W)^\perp$.

   (b) Prove that $\dim(W) - \dim(U\cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp)$.

3. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Let λ_1 be the maximum of the eigenvalues of B. For $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$, using the usual 2-norm $\|\mathbf{x}\|_2$, define the Raleigh quotient $\rho_B(\mathbf{x})$ for B by

$$\rho_B(\mathbf{x}) = \frac{(B\mathbf{x},\mathbf{x})}{(\mathbf{x},\mathbf{x})} = \frac{\mathbf{x}^t B\mathbf{x}}{\|\mathbf{x}\|_2^2}$$

   Prove the following:

   (a) If B and λ_1 are defined as above, prove that $\lambda_1 = \max\{\rho_B(\mathbf{x}) : \ \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_2 = 1\,\}$

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with largest singular value σ_1. If

$$\|A\|_2 = \max\{\|A\mathbf{x}\|_2 : \ \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_2 = 1\,\}$$

   show that $\|A\|_2 = \sigma_1\,.$

4. Let $A = A^{(1)}$ be strictly column diagonally dominant. After one step of Gauss elimination $A^{(1)}$ is reduced to

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12}\cdots a_{1n} \\ 0 & A_s^{(2)} \end{pmatrix}.$$

   Show that $A_s^{(2)} \in \mathbb{R}^{(n-1)\times(n-1)}$ is also strictly column diagonally dominant. (Therefore, if LU decomposition with partial pivoting is applied to A, no row interchanges take place.)

5. Let $A_1, A_2, \ldots, A_k \in \mathbb{R}^{n\times n}$ such that A_1 has n distinct eigenvalues. Prove that there exists an invertible $P \in \mathbb{R}^{n\times n}$ such that $P^{-1}A_jP$ is a diagonal matrix for each $1 \le j \le k$ if and only if $A_iA_j = A_jA_i$ for all $1 \le i, j \le k$.

6. Let V be a finite dimensional inner product space and $W \subset V$ a subspace. For every $v \in V$ there is a unique decomposition $v = w + w'$ with $w \in W$ and $w' \in W^{\perp}$. Define a map $T : V \to V$ by $Tv = w - w'$. Prove that T is a unitary and self-adjoint operator.

7. Let A be an $n \times n$ real matrix of full rank with eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ and let X be a matrix that diagonalizes A, i.e. $X^{-1}AX = D$ where D is a diagonal matrix. If $A' = A + E$ and λ' is an eigenvalue of A', prove that

$$\min_{1\le i\le n} |\lambda' - \lambda_i| \le \kappa_2(X)\|E\|_2$$

where $\kappa_2(X)$ is the 2-norm condition number of X.

8. Given the data $(0, 1), (3, 4), (6, 5)$. Use a QR factorization technique to find the best least squares fit by a linear function.