UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics
Joint Program Exam in Linear Algebra and Numerical Linear Algebra

September 2011

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen

1. Let an $n \times n$ matrix A be strictly column diagonally dominant, i.e.,

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ji}|, \qquad i = 1, \ldots n.$$

Show that A is nonsingular.

2. Suppose that A is an $n \times n$ matrix with 1 on the main diagonal, 2 on the first superdiagonal, and 0 everywhere else.

   (a) What are the eigenvalues, determinant, and rank of A?

   (b) What is A^{-1}?

   (c) Prove that $\sigma_1 \leq 3$, where σ_1 is the largest singular value of A.

3. Let A be an $n \times n$ matrix, and let a_j denote its j-th column. Prove the following inequality:

$$|\det A| \leq \prod_{j=1}^{n} \|a_j\|_2$$

4. Consider a 4×4 matrix defined by

$$A_t = \begin{bmatrix} t & 1 & 1 & 1 \\ 1 & t & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{bmatrix}$$

(a) Find all the values of t for which this matrix is singular.
(b) Find $\operatorname{rank}(A_t)$ for the values of t from part (a).

5. (a) For which value(s) of x are the matrices

$$A = \begin{bmatrix} 1 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

similar?
(b) Prove that whenever these two matrices are similar, then they are orthogonally equivalent.

6. Let A be a real symmetric and upper Hessenberg matrix of size $n \times n$. Show that
$$|\lambda - a_{n,n}| \leq |a_{n,n-1}|$$
for some eigenvalue λ of A. (The meaning is that if $a_{n,n-1}$ is small, then $a_{n,n}$ is a good approximation to an eigenvalue of A.)

7. Suppose the 2-condition number of a full rank rectangular matrix A of size $m \times n$ (with $m > n$) is defined by
$$\kappa_2(A) := \frac{\sup_{\|x\|_2=1} \|Ax\|_2}{\inf_{\|x\|_2=1} \|Ax\|_2}.$$
Prove that
$$\min \left\{ \frac{\|A - A_d\|_2}{\|A\|_2} : \; A_d \text{ is rank deficient} \right\} = \frac{1}{\kappa_2(A)}.$$

8. Let $A(t) = (a_{ij}(t))$ be a square $n \times n$ matrix whose components are differentiable functions of t (i.e., $\frac{d}{dt} a_{ij}(t)$ exists for all $i, j = 1, \ldots, n$).
(a) Prove that
$$\frac{d}{dt} \det A(t) =$$
$$= \sum_{k=1}^{n} \det \begin{bmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,k-1}(t) & \frac{d}{dt} a_{1,k}(t) & a_{1,k+1}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,k-1}(t) & \frac{d}{dt} a_{2,k}(t) & a_{2,k+1}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i,1}(t) & a_{i,2}(t) & \cdots & a_{i,k-1}(t) & \frac{d}{dt} a_{i,k}(t) & a_{i,k+1}(t) & \cdots & a_{i,n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,k-1}(t) & \frac{d}{dt} a_{n,k}(t) & a_{n,k+1}(t) & \cdots & a_{n,n}(t) \end{bmatrix}$$
(Hint: you may want to use the multilinearity of the determinant when taking the derivative).
(b) Use part (a) to show that, if I_n is the identity matrix of size n, B is a constant matrix, and $\mathrm{tr}(B)$ denotes the trace of B, then
$$\frac{d}{dt} \det(I_n + tB)|_{t=0} = \mathrm{tr}(B).$$
Note: we first take the derivative of $\det(I_n + tB)$ with respect to t and then substitute $t = 0$.