University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2011

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. (a) Let $T \in \mathcal{L}(\mathbb{R}^n)$ be a linear map with $(T(\mathbf{x}), \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, where $(\cdot, \cdot)$ denotes the inner-product on $\mathbb{R}^n$. Show that $T^* = -T$.

   (b) Prove that if there exists a linear map $V \to W$ whose null space and range are both finite dimensional, then V is finite dimensional.

2. Let $\alpha = \{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ be an arbitrary ordered basis for $\mathbb{R}^5$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Define $T : \mathbb{R}^5 \to \mathbb{R}^5$ by $[T(\mathbf{v})]_\alpha = A[\mathbf{v}]_\alpha$ for all $\mathbf{v} \in \mathbb{R}^5$, where $[\mathbf{v}]_\alpha$ denotes the coordinate representation of $\mathbf{v}$ relative to the basis α.

   (a) Compute the eigenvalues of T and both the minimal and characteristic polynomial of T.

   (b) Find the Jordan form for T.

3. Let A be an $n \times n$ complex matrix. Define $M = \frac{1}{2}(A + A^*)$ and $N = \frac{1}{2}(A - A^*)$. Prove that A is normal if every eigenvector of M is also an eigenvector of N.

4. Let $A \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$ be a unit vector in the 2-norm, $\tau \in \mathbb{R}$ and $\mathbf{r} = A\mathbf{x} - \tau\mathbf{x}$.

   (a) Show that τ is an eigenvalue of a matrix $A + E$, where $\|E\|_2 \leq \|\mathbf{r}\|_2$.

   (b) Assuming in addition that A is symmetric, show that there exists an eigenvalue λ of A such that $|\lambda - \tau| \leq \|\mathbf{r}\|_2$.

5. Prove that any Givens rotator matrix in $\mathbb{R}^n$ is a product of two Householder reflector matrices. Can a Householder reflector matrix be a product of Givens rotator matrices?

6. Suppose an $m \times n$ matrix has the form $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a nonsingular matrix of dimension $n \times n$ and A_2 is an arbitrary matrix of dimension $(m-n) \times n$ with $m > n$. Let $A^\dagger$ be the pseudo inverse of A defined as $(A^*A)^{-1}A^*$. Prove that $\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$.

7. Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix. Let the nonzero singular values of A be $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$. Prove the following:

   (a) The $\text{rank}(A)$ is r.

   (b) $\|A\|_2 = \sigma_1$, where $\|A\|_2$ is the $2-$ norm of A.

   (c) $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_2$, where $\|\cdot\|_F$ is the Frobenius norm of A.

8. Let $S \in \mathbb{C}^{m\times m}$ be skew-Hermitian, i.e., $S^* = -S$. Show the following:

   (a) The eigenvalues of S are purely imaginary.

   (b) The matrix $I - S$ is invertible.

   (c) The matrix $Q = (I - S)^{-1}(I + S)$ is unitary.