UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN
APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May 1997

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will accrue from answering 5 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 5 problems.

1. Let $P_2(\mathbf{R})$ be the set of polynomials of degree less than or equal to 2, defined over the real line, and define

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$$

according to

$$T(p) = q$$

where

$$q(x) = (1 - x)p'(x)$$

   (a) Find a basis for the range and the null space of this transformation. Is it invertible?

   (b) Find the characteristic polynomial, minimal polynomial and Jordan canonical form of T.

2. Let V be the vector space of all complex valued polynomials defined over the half line $[0, \infty)$.

   (a) Show that

$$\langle f, g \rangle := \int_0^\infty f(x)\overline{g(x)}e^{-x}\, dx$$

   is a complex inner product on V.

   (b) Find an orthonormal set $\{f_0, f_1\}$ in V such that $\text{span}\{e_0, e_1\} = \text{span}\{f_0, f_1\}$, where $e_0(x) = 1$ and $e_1(x) = x$.

3. Suppose that A is a complex normal matrix. Prove that

   (a) A and A^* have the same eigenvectors;

   (b) if x and y are two eigenvectors of A corresponding to distinct eigenvalues, then x and y are orthogonal;

   (c) if A is also upper-triangular, then A must be diagonal.

4. (a) Give a definition of the condition number, $K(A)$, of a matrix A with respect to the infinity norm.

   (b) Compute the condition number of

$$A = \begin{pmatrix} 1 & 2 \\ 1.01 & 2 \end{pmatrix}$$

(c) Show that if B is singular, then

$$\frac{1}{K(A)} \leq \frac{\|A-B\|}{\|A\|}$$

(d) Use (c) to estimate $K(A)$, where A is the matrix given in (b), and compare to the solution obtained in (b).

5. (a) Give an explanation of what is meant by the least squares solution of $Ax = b$, where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$.

   (b) Find the least squares solution of the system

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

   (c) Also compute the norm of the minimal residual vector.

6. Let $A \in \mathbf{R}^{n\times n}$ be given, and assume that all leading principal submatrices of A are nonsingular. Show that there exists a unique upper triangular matrix U and a unique unit lower triangular matrix L, such that $A = LU$. If A is nonsingular, but not all the leading principal submatrices are nonsingular, what is the result now? (You don't have to prove this one, just explain it.)

7. Let $A \in \mathbf{R}^{n\times n}$ be given, symmetric, and assume that the eigenvalues of A satisfy

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| \geq |\lambda_n|.$$

   Let $z \in \mathbf{R^n}$ be given. Under what conditions on z does the following hold, theoretically? (Be sure to actually show that it holds!)

$$\lim_{k\to\infty} \frac{z^T A^{k+1} z}{z^T A^k z} = \lambda_1$$

   Under what conditions on z does this hold, *as a practical matter?* Explain fully for full credit.