EGR 265, Math Tools for Engineering Problem Solving
May 1, 2009, 1:30pm to 4:00pm

Name (Print last name first): ....

Student ID Number: ....

Final Exam

Problem 1 (8 points)

Find an explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{2x}{y}, \quad y(1) = 2.$$

Problem 2 (8 points)

A population of bacteria grows proportional to the number of bacteria present at time t. Suppose that the initial population is 100 and that the population after 2 hours has grown to 150.
(a) Find the growth rate k of the population.
(b) How long does it take the population to double in size?
Note: Write your answers in terms of natural logarithms, which do not need to be evaluated.

Problem 3 (14 points)

Consider the second order differential equation

$$y'' - y' - 2y = \cos x + \sin x. \tag{1}$$

(a) Find the general solution of the homogeneous equation corresponding to (1).
(b) Find a particular solution of the inhomogeneous equation (1).
(c) Solve the initial value problem given by (1) and initial conditions $y(0) = 0$, $y'(0) = 0$.

Problem 4 (12 points)

A mass of 4 kg stretches a spring by 40 cm.
(a) Find the spring constant k, assuming that $g = 10$ m/s^2.
(b) Find the equation of motion of the mass if it is released 10 cm above the equilibrium position at a downward velocity of 2 m/s.
(c) What is the amplitude at which the mass oscillates?
(d) How many full oscillations will the mass have completed in 4π seconds?

Problem 5 (10 points)

(a) Find the gradient of $f(x, y) = \frac{x+y}{x-y}$.
(b) Evaluate the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction of the vector $\mathbf{i} - \mathbf{j}$.
(c) Find a unit vector in the direction of steepest increase of $f(x, y)$ at the point $(2, 1)$. Also find the rate of increase in this direction.

Problem 6 (8 points)

Determine the equation of the tangent plane to the level surface $\ln x + \cos y + xz^3 = 8$ at the point $(1, \pi/2, 2)$.

Problem 7 (8 points)

Find the line integral

$$\int_C \sqrt{1 + 4y^2}\, ds,$$

where C is the curve parameterized by $x = \ln t$, $y = t^2/2$, $1 \leq t \leq 2$.

Problem 8 (12 points)

(a) Show that the force field $F(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$ is conservative and find a potential function $\phi(x, y)$ for it.
(b) Find the work done by the force field F from part (a) along the curve $x(t) = \cos t$, $y = \sin t$, $0 \leq t \leq \pi/2$.

Problem 9 (10 points)

A lamina of constant density $\rho(x, y) = 1$ lies between the lines $y = 0$, $x = 0$ and $y = 2 - 2x$.
(a) Find the lamina's mass without doing an integration.
(b) Find the center of mass of the lamina.

Problem 10 (10 points)

Find the double integral of the function $f(x, y) = e^{x^2+y^2}$ over the region in the xy-plane which is bounded by the circles $r = 1$ and $r = 2$ and lies above the x-axis.

Problem 11 (6 points Bonus)

Find the volume of the infinite solid above the xy-plane and under the two-dimensional bell curve $f(x, y) = e^{-x^2-y^2}$. Do this by evaluating the "double improper integral"

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA$$

using the following steps:
(a) Find the double integral of $e^{-x^2-y^2}$ over a disk of radius R.
(b) In the result from part (a) take the limit $R \to \infty$.