Mathematical Analysis Qualifying Exam
May 11, 2026

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.

1. Let $\{a_n\}_{n=1}^{\infty}$ be a positive sequence. Show that

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

2. Show that if f is a differentiable function $f : [0,1] \to [0,1]$, such that $f'(x) < 1$ for all $x \in [0,1]$, then there is exactly one $x \in [0,1]$ such that $f(x) = x$.

3. Let $f : [0,\infty) \to \mathbb{R}, g : [0,1] \to \mathbb{R}$ be continuous functions, so that $\lim_{x\to\infty} f(x) = A$ and $\int_0^1 g(x)dx = B$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n f(x) g\left(\frac{x}{n}\right) dx = AB.$$

Remark: We are looking for a direct proof. A solution through the Lebesgue dominated convergence theorem is not acceptable.

4. Suppose $f : [0,1] \to \mathbb{R}$ is a bounded function, so that for every $\delta \in (0,1)$, f is Riemann integrable on $[\delta, 1]$. Prove that f is Riemann integrable on $[0,1]$ and

$$\int_0^1 f(x)dx = \lim_{\delta\to 0+} \int_\delta^1 f(x)dx.$$

Hint: First show that f is Riemann integrable on $[0,1]$.

5. Let $f : [a,b] \to \mathbf{R}$ be a continuous function. Show that

$$\lim_{p\to+\infty} \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} = \max_{x\in[a,b]} |f(x)|.$$

6. Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable functiton on (a,b). We say that the function is convex, if $f''(x) \geq 0, a < x < b$. Prove that f is convex, if and only if, for all $x, y \in (a,b)$,

$$f(y) \geq f(x) + f'(x)(y - x).$$

7. Let A and B be nonempty, disjoint subsets of a metric space (X,d), so that A is closed and B is compact. Show that

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\} > 0.$$