EGR/MA265, Math Tools for Engineering Problem Solving
Final Exam, 2013

Name and section: _____

Instructors name: _____

1. **Do not open this exam until you are told to do so.**

2. This exam has 14 pages including this cover. There are 10 questions, for a total of 100 points plus 5 bonus points. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

6. Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.

8. **Turn off all cell phones**, and remove all headphones.

Do not write in the table below.

Question	Points	Score
1	8	
2	10	
3	14	
4	10	
5	10	
6	8	
7	8	
8	12	
9	12	
10	8	
Total:	100	

1. [8 points] Find an explicit solution of the initial value problem

$$\frac{y'}{(1+x^2)^5} = \frac{x}{y^2}, \quad y(0) = 0.$$

2. [10 points] Over two years after the initial disaster at Japan's Fukushima Daiichi nuclear power facility, the plant is leaking 300 tons of radioactive water into the surrounding ocean. The radioactive water contains about 30 trillion units (in Becquerels) of radioactive substance **tritium**. It is known that tritium has a half-life of 12 years.

   (a) State the initial value problem, i.e., a differential equation with an initial condition, which governs the amount of tritium in the radioactive water.

   (b) Solve this initial value problem.

   (c) Determine the decay rate of tritium.

   (d) How much of tritium (in trillion units) is left after 24 years?

   (e) How long does it take for the tritium to decay to 20 percent of the original amount?

   Note : You can write your answers in terms of natural logarithms.

3. [14 points] Consider the second order differential equation

$$y'' - 4y' + 4y = x + 1. \tag{1}$$

(a) Find the general solution of the homogeneous equation corresponding to (1).

(b) Find a particular solution of the inhomogeneous equation (1).

(c) Solve the initial value problem given by (1) and initial conditions $y(0) = 0$, $y'(0) = 0$.

4. [10 points] A mass of 40 kg stretches an undamped spring by 40 cm. Assume that $g = 10$ m/s^2.

    (a) Find the spring constant k, including its correct unit.

    (b) Set up the second order differential equation which governs the motion of the spring-mass system, choosing the x-axis to be oriented downwards. Find the general solution of this equation.

    (c) Find the unique solution of the equation if the mass is released from the equilibrium position at an upward velocity of 10 cm/s.

(d) [5 points (bonus)] Suppose that in the above problem a damping force proportional to $\beta = 1000$ kg/sec times the instantaneous velocity is added to the spring-mass system. Does the system become underdamped, critically damped or overdamped?

5. [10 points] (a) Find the gradient of $f(x, y) = \frac{1}{2x-y}$.

(b) Evaluate the directional derivative of $f(x, y)$ at the point with coordinates $(1, 1)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$.

(c) Find a unit vector in the direction of steepest decrease of $f(x, y)$ at the point $(1, 1)$.

6. [8 points] Determine the equation of the normal line to the graph of $z = x^2 - xy + 3y^2$ through the point $(2, 1, 5)$.

7. [8 points] Find the work done by the force field

$$\mathbf{F}(x, y) = x\sqrt{1 + 2y}\ \mathbf{i} + 2xy\ \mathbf{j}$$

along the graph of $y = \frac{1}{2}x^2$, $0 \le x \le \sqrt{3}$.

8. [12 points] (a) Verify that the force field $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + (\cos(y) + x^2 y)\,\mathbf{j}$ is conservative.

(b) Find a potential function $\phi(x, y)$ for $\mathbf{F}(x, y)$.

(c) Find the work done by the force field $\mathbf{F}(x, y)$ along the graph of the function

$$y = (x^2 - 1)^8, \quad -1 \le x \le 1.$$

9. [12 points] A lamina of constant density $\rho(x, y) = 1$ is bounded by the curves $x = 0$, $y = 0$ and $y = 2 - x$.

   (a) Find the moment of inertia I_x with respect to the x-axis.

   (b) Find the moment of inertia I_y with respect to the y-axis.

10. [8 points] Let R be the upper half of a disk centered at the origin with radius 2. Use polar coordinates to find the double integral of the function $f(x, y) = 2e^{x^2 + y^2}$ over region R.