EGR 265-6D, Math Tools for Engineering Problem Solving
December 7, 2009, 1:30pm to 4:00pm

Name (Print last name first): . . . .

Student ID Number: . . . .

Final Exam

Problem 1 (8 points)

Find an explicit solution of the initial value problem

$$(1 + x^2)yy' = x, \quad y(0) = 2.$$

Problem 2 (8 points)

A radioactive isotope has a half-life of 10 years.
(a) Find its decay rate k (which should be a negative number).
(b) If the initial amount of the isotope is 1 gram, how much of it is left after 5 years?
(c) How long does it take for the isotope to decay to 10 percent of its original amount?
Note: Write your answers in terms of natural logarithms, which do not need to be evaluated.

Problem 3 (14 points)

Consider the second order differential equation

$$y'' + y' - 2y = 2e^x. \tag{1}$$

(a) Find the general solution of the homogeneous equation corresponding to (1).
(b) Find a particular solution of the inhomogeneous equation (1).
(c) Solve the initial value problem given by (1) and initial conditions $y(0) = 0$, $y'(0) = 0$.

Problem 4 (12 points)

A mass of 10 kg stretches a spring by 50 cm. Include the correct units in all your answers below.
(a) Find the spring constant k, assuming that $g = 10$ m/s^2.
(b) What is the frequency at which the mass oscillates?
(c) Find the equation of motion of the mass if it is released from rest at a position 20 cm below the equilibrium position (choose the positive x-axis to be oriented downward).
(d) Find the first positive time at which the mass passes through the equilibrium position.

Problem 5 (10 points)

(a) Find the gradient of $f(x, y) = \sqrt{x^2 + y^3}$.
(b) Evaluate the directional derivative of $f(x, y)$ at the point $P(1, 2)$ in the direction from P to the point $Q(3, 3)$.
(c) Find a unit vector in the direction of steepest decrease of $f(x, y)$ at the point $(1, 2)$. Also find the rate of increase in this direction.

Problem 6 (8 points)

Determine parametric equations of the normal line to the graph of $z = \frac{x}{x+y}$ at the point $(1, -2, -1)$.

Problem 7 (8 points)

Find the line integral

$$\int_C x^2 y \, ds,$$

where C is a quarter of a unit circle centered at the origin and contained in the first quadrant, starting at $(1, 0)$ and ending at $(0, 1)$.

Problem 8 (12 points)

(a) Show that the force field $F(x,y) = (4e^y - 2ye^x)\mathbf{i} + (4xe^y - 2e^x)\mathbf{j}$ is conservative and find a potential function $\phi(x,y)$ for it.
(b) Find the work done by the force field F from part (a) along the curve $x(t) = t^2$, $y = t^3$, $0 \le t \le 1$.

Problem 9 (10 points)

A lamina of constant density $\rho(x, y) = 1$ is bounded by the curves $y = x^2$ and $y = 1$.
(a) Find the lamina's mass.
(b) Find the lamina's centroid. Use geometric considerations to simplify your work.

Problem 10 (10 points)

Find the double integral of the function $f(x, y) = e^{\sqrt{x^2+y^2}}$ over the region in the first quadrant which is bounded by the circles $r = 1$ and $r = 2$.