FALL 2009 — MA 227-6B — TEST 3
NOVEMBER 18, 2009

Name: ____

There are 7 problems on this test. The number of points to earn is indicated for each problem. Partial credit is awarded where appropriate. Your solution must include enough detail to justify any conclusions you reach in answering the question. Points will be awarded for the correct reasoning.

(1) (21 points) Find the local maxima, minima, and saddle points of the function $g(x, y) = x^2 + 2y^2 + x^2y$.

Solution:

Local extrema and saddle points occur at critical points, i.e., points where the gradient is zero. The gradient of g is

$$\nabla g = \langle 2x(1 + y), 4y + x^2\rangle.$$

The first component is zero either if $x = 0$ or else if $y = -1$.

In the first case, when $x = 0$, the second component is $4y$ which is zero when $y = 0$. Hence $(0, 0)$ is a critical point.

In the second case, when $y = -1$, the second component is $x^2 - 4$ which is zero when $x = \pm 2$. Hence $(2, -1)$ and $(-2, -1)$ are critical points.

The nature of a critical point is (often) decided by the second derivative test. We get $g_{xx} = 2 + 2y$, $g_{xy} = g_{yx} = 2x$, and $g_{yy} = 4$. Hence

$$D(x, y) = g_{xx}g_{yy} - g_{xy}^2 = 8(1 + y) - 4x^2.$$

Since $D(0, 0) = 8 > 0$ the point $(0, 0)$ is an extremum. Since $g_{xx}(0, 0) = 2 > 0$ (note that also $g_{yy} = 4 > 0$) the point $(0, 0)$ is a minimum.

Since $D(\pm 2, 1) = -16 < 0$ the points $(2, -1)$ and $(-2, -1)$ are saddle points.

(2) (21 points) Use Lagrange multipliers to find maximum and minimum values of the function $f(x, y) = 2x + y + 5$ on the circle $x^2 + y^2 = 1$. Where do they occur?

Solution:

Only points on the circle are to be considered. This constraint is expressed as $g(x, y) = 0$ where $g(x, y) = x^2 + y^2 - 1$. The method of Lagrange multipliers requires to find the solutions of the following system of algebraic equations:

$$\nabla f = \lambda \nabla g,$$
$$g = 0.$$

In this case $\nabla f = \langle 2, 1\rangle$ and $\nabla g = \langle 2x, 2y\rangle$ so that the system is

$$2 = 2\lambda x$$
$$1 = 2\lambda y$$
$$1 = x^2 + y^2.$$

From the first equation (for instance) we get that λ can not be zero (otherwise the first equation would be $2 = 0$ which is absurd). Hence $x = 2/(2\lambda)$ and $y = 1/(2\lambda)$. Therefore $x = 2y$. Using this in the third equation gives

$$1 = (2y)^2 + y^2 = 5y^2.$$

There are two solutions: $y_1 = 1/\sqrt{5}$ and $y_2 = -1/\sqrt{5}$. Recalling $x = 2y$ gives two points to consider: $P_1 = (2, 1)/\sqrt{5}$ and $P_2 = -(2, 1)/\sqrt{5}$. The value of f at P_1 is $5/\sqrt{5}+5$ and the value of f at P_2 is $-5/\sqrt{5}+5$. The former is the larger of the two. Hence the maximum occurs at P_1 and the minimum at P_2.

(3) (21 points) Evaluate the integral

$$\iint_D (2x+y)dA$$

where D is the region bounded by the curves $y = x$ and $y = x^2$.

Solution:

The graphs of $y = x$ and $y = x^2$ intersect at $(0,0)$ and at $(1,1)$. The graph of $y = x^2$ lies below the one for $y = x$ when $0 \le x \le 1$. Thus, for fixed x between 0 and 1 the variable y varies between x^2 and x. The iterated integral reads

$$\int_0^1 \int_{x^2}^x (2x+y)dydx.$$

We compute

$$\begin{aligned}
\int_0^1 \int_{x^2}^x (2x+y)dydx &= \int_0^1 (2xy + \frac{1}{2}y^2)\Big|_{y=x^2}^{y=x} dx \\
&= \int_0^1 ((2x^2 + \frac{1}{2}x^2) - (2x^3 + \frac{1}{2}x^4))dx \\
&= \int_0^1 (\frac{5}{2}x^2 - 2x^3 - \frac{1}{2}x^4)dx \\
&= (\frac{5}{6}x^3 - \frac{1}{2}x^4 - \frac{1}{10}x^5)\Big|_{x=0}^{x=1} \\
&= \frac{5}{6} - \frac{1}{2} - \frac{1}{10} \\
&= \frac{7}{30}.
\end{aligned}$$

(4) (13 points) Use polar coordinates to evaluate the integral $\iint_D xy^2 dA$ where D is the left half of the disk of radius 3 centered at the origin.

Solution:

For polar coordinates we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The points in the left half plane have values of θ between $\pi/2$ and $3\pi/2$. The points in the disk of radius 3 centered at the origin have values of r between 0 and 3. We must not forget that $dA = r dr d\theta$. Hence

$$\iint_D xy^2 dA = \int_{\pi/2}^{3\pi/2} \int_0^3 r\cos(\theta)(r\sin(\theta))^2 r dr d\theta$$
$$= \int_{\pi/2}^{3\pi/2} \cos(\theta)(\sin(\theta))^2 \int_0^3 r^4 dr d\theta = \frac{r^5}{5}\Big|_{r=0}^{r=3} \int_{\pi/2}^{3\pi/2} \cos(\theta)(\sin(\theta))^2 d\theta.$$

To evaluate the θ-integral we substitute $u = \sin(\theta)$, $du = \cos(\theta)d\theta$. $\sin(\pi/2) = 1$ will be the lower limit of the integral while $\sin(3\pi/2) = -1$ will be the upper limit. Hence

$$\int_{\pi/2}^{3\pi/2} \cos(\theta)(\sin(\theta))^2 d\theta = \int_1^{-1} u^2 du = \frac{u^3}{3}\Big|_{u=1}^{u=-1} = -\frac{2}{3}.$$

Together with the previous result we get

$$\iint_D xy^2 dA = -\frac{3^5}{5}\frac{2}{3} = -2\frac{3^4}{5} = -\frac{162}{5}.$$

(5) (13 points) Find the absolute maximum and minimum as well as the points where they occur for the function $f(x, y) = 3 + xy$ over the disk of radius $\sqrt{8}$ centered at the origin.

Solution:

We have to investigate both the interior and the boundary of the disk. For the interior we find the critical points of f, i.e., the zeros of ∇f. Since $\nabla f = \langle y, x\rangle$ there is precisely one critical point, the origin. Note that $f(0, 0) = 3$.

The boundary of the disk is described by the constraint equation $g(x, y) = 0$ where $g(x, y) = x^2 + y^2 - 8$. Here we use Lagrange multipliers. This method requires to find the solutions of the following system of algebraic equations:

$$\begin{aligned} \nabla f &= \lambda \nabla g, \\ g &= 0. \end{aligned}$$

In this case $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x, 2y\rangle$ so that the system is

$$\begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ 8 &= x^2 + y^2. \end{aligned}$$

From the first two equations we get $x = 4\lambda^2 x$. If $x = 0$ then $y = 0$ and $0 = 8$ so that x cannot be zero. Therefore we get $4\lambda^2 = 1$ and 2λ is either 1 or -1.

First let $2\lambda = 1$, so that $x = y$. Then the third equation is $2x^2 = 8$ which holds when $x = 2$ or $x = -2$. We have two points to consider: $(2, 2)$ and $(-2, -2)$. At both points f assumes the value 7.

Now let $2\lambda = -1$, so that $x = -y$. Then the third equation is again $2x^2 = 8$. Now the two points are $(2, -2)$ and $(-2, 2)$. At both of these f assumes the value -1.

Comparing the three values -1, 3, and 7 we see that the minimum is -1 attained at $(2, -2)$ and $(-2, 2)$ and the maximum is 7 attained at $(2, 2)$ and $(-2, -2)$.

(6) (7 points) Compute the centroid of the solid E which is bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3x^2 - 3y^2$.

Solution:

E is symmetric with respect to rotation about the z-axis. The centroid lies therefore on the z-axis, i.e., $\bar{x} = \bar{y} = 0$. The value $\bar{z}$ is given by the integral

$$\iiint_E z dV.$$

The points of E satisfy $x^2 + y^2 \leq z \leq 4 - 3x^2 - 3y^2$. Therefore we need to consider only points (x, y) where $x^2 + y^2 \leq 4 - 3x^2 - 3y^2$. Using equality here gives us the boundary of that region, but $x^2 + y^2 = 4 - 3x^2 - 3y^2$ precisely when $x^2 + y^2 = 1$, a circle of radius 1 centered at the origin.

Using cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$ we get

$$\iiint_E z dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{4-3r^2} zrdzdrd\theta$$

(remember the Jacobian for cylindrical coordinates). Hence

$$\iiint_E z dV = 2\pi \int_0^1 \frac{1}{2} r z^2 \Big|_{z=r^2}^{z=4-3r^2} dr$$

$$= \pi \int_0^1 (r(4-3r^2)^2 - r^5) dr = \pi \int_0^1 (16r - 24r^3 + 8r^5) dr$$

$$= \pi(8 - 6 + \frac{8}{6}) = \frac{10\pi}{3}.$$

(7) (4 points) The ellipse E defined $4x^2 + 9y^2 \leq 36$ can be transformed into a circle by a change of variables. Perform such a change of variables to find the area of the ellipse.

Solution:

Dividing by 36 we may express ellipse as

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 \leq 1.$$

Thus we substitute $u = x/3$ and $v = y/2$, i.e., $x = 3u$ and $y = 2v$ to get the disk $u^2 + v^2 \leq 1$. This transformation has Jacobian

$$J = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

Then

$$\text{Area}(E) = \iint_E 1 dA(x,y) = \iint_D J dA(u,v) = J \iint_D 1 dA(u,v) = J\text{Area}(D)$$

where D is the disk $u^2 + v^2 \leq 1$ which has area π. Hence the area of E is 6π.