FALL 2007 — MA 227-6B — TEST 4
DECEMBER 5, 2007

Name: ______________________________

1. Part I

There are 6 problems in Part 1, each worth 4 points. Place your answer on the line to the right of the question. Only your answer on the answer line will be graded.

(1) Compute div $\mathbf{F}$ $(= \nabla \cdot \mathbf{F})$ when $\mathbf{F}(x, y, z) = \langle x + y, \mathrm{e}^{xyz}, 5\cos(xy)\rangle$.

Solution: $1 + xz\mathrm{e}^{xyz}$, a scalar!

(2) Find the curl of the vector field $\mathbf{F}(x, y, z) = \langle 3xyz, 0, -x^2y\rangle$.

Solution: $\langle -x^2, 5xy, -3xz\rangle$, a vector!

(3) Compute grad f $(= \nabla f)$ when $f(x, y, z) = x^2 + y + z$.

Solution: $\langle 2x, 1, 1\rangle$

(4) Find a parametrization for the cone $z = 2\sqrt{x^2 + y^2}$.

Solution: $\langle x, y, 2\sqrt{x^2 + y^2}\rangle$ or $\langle z\cos(\theta)/2, z\sin(\theta)/2, z\rangle$

(5) Find a function f such that $(\nabla f)(x, y) = \langle y^2, 2xy + 1\rangle$.

Solution: $xy^2 + y$

(6) Evaluate the line integral $\int_C 3ds$ when C is the semicircle $\langle \cos(t), \sin(t)\rangle$, $0 \le t \le \pi$.

Solution: 3π

2. Part II

There are 3 problems in Part 2, each worth 12 points. On Part 2 problems partial credit is awarded where appropriate. Your solution must include enough detail to justify any conclusions you reach in answering the question.

(1) Let C be the boundary of the unit square (with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$) oriented counterclockwise. Evaluate

$$\int_C (2y\,dx + (x^2 - x)\,dy)$$

by two methods: directly as a line integral and using Green's Theorem.

Solution:

1. Using Green's theorem:
$Q_x - P_y = 2x - 1 - 2 = 2x - 3$ needs to be integrated over the unit square, i.e., $\int_0^1 \int_0^1 (2x-3)dydx = -2$.

2. Directly:
There are four pieces which make up the curve C: On the vertical ones $x^2 - x = 0$ and $x'(t) = 0$. These contribute nothing to the integral. On the bottom one $y = 0$ and $y'(t) = 0$. It also contributes nothing. On the top one $x(t) = 1 - t$, $y(t) = 1$ so that we get $\int_0^1 2(-dt) = -2$.

(2) Find the work done by the force field $\mathbf{F} = 3x\,\mathbf{i} + (y+9)\,\mathbf{j}$ on a particle that moves along a line segment from the point $(-1,2)$ to the point $(2,3)$.

First Solution:

Work is the given by the line integral $\int_C \mathbf{F}\cdot d\,\mathbf{r}$.

$\mathbf{r}(t) = \langle 3t-1, t+2\rangle$ where $0 \le t \le 1$. Therefore $\mathbf{r}'(t) = \langle 3, 1\rangle$.

$\mathbf{F}(x,y) = \langle 3x, y+9\rangle$, $\mathbf{F}(\,\mathbf{r}(t)) = \langle 9t-3, t+11\rangle$, $\mathbf{F}(\,\mathbf{r}(t))\cdot\mathbf{r}'(t) = 27t - 9 + t + 11 = 28t + 2$.

$\int_C \mathbf{F}\cdot d\,\mathbf{r} = \int_0^1 (28t+2)dt = 16.$

Second Solution:

F is conservative with potential $f(x,y) = 3x^2/2 + y^2/2 + 9y$. The line integral equals then $f(2,3) - f(-1,2) = 16$.

(3) Find the surface area of that part of the paraboloid $3x^2+3y^2+z=12$ that lies above the x-y-plane.

Solution:

We can use the polar coordinate r and θ as parameters:

$$\mathbf{r}(r,\theta)=\langle r\cos(\theta), r\sin(\theta), 12-3r^2\rangle$$

Here $0\leq\theta\leq 2\pi$. r may be as small as zero and becomes largest when $z=0$ where $3r^2=12$, i.e, $r^2\leq 4$ and $r\leq 2$.

Compute the tangent vectors $\mathbf{r}_r$ and $\mathbf{r}_\theta$ and form their cross product:

$$\begin{aligned}&\langle\cos(\theta),\sin(\theta),-6r\rangle\times\langle -r\sin(\theta), r\cos(\theta), 0\rangle\\ =&\langle 6r^2\cos(\theta), 6r^2\sin(\theta), r\rangle.\end{aligned}$$

The length of this normal vector is $\sqrt{36r^4+r^2}=r\sqrt{36r^2+1}$.

Surface area is given by

$$\iint_S dS=\iint_D|\mathbf{r}_r\times\mathbf{r}_\theta|d(r,\theta)=\int_0^2\int_0^{2\pi} r\sqrt{36r^2+1}d\theta dr.$$

The θ-integration gives a factor 2π and for the r-integration we substitute $u=36r^2+1$, $du=72r\,dr$. The new limits are given by $1\leq u\leq 145$. Thus the area is

$$\frac{2\pi}{72}\int_1^{145}\sqrt{u}du=\frac{\pi}{36}\frac{2}{3}(145^{3/2}-1)=\frac{\pi}{54}(145^{3/2}-1).$$