FALL 2006 — MA 227-8B — TEST 4

Name: _____

1. PART I

There are 4 problems in Part I, each worth 4 points. Place your answer on the line below the question. In Part I, there is no need to show your work, since only your answer on the answer line will be graded.

(1) Find the gradient vector field of $f(x, y) = \sin(x - y^2)$.

Answer:
$$\operatorname{grad} f = \langle \cos(x - y^2), -2y\cos(x - y^2) \rangle$$

(2) Find the divergence of the vector field $\mathbf{F}(x, y, z) = \langle y\sqrt{x}, z\ln y, \frac{z^2}{y} \rangle$.

Answer:
$$\operatorname{div} \mathbf{F} = \frac{y}{2\sqrt{x}} + \frac{3z}{y}$$

(3) Find the curl of the vector field $\mathbf{F}(x, y, z) = \langle xyz, xz^2, 0 \rangle$.

Answer:
$$\operatorname{curl} \mathbf{F} = \langle -2xz, xy, z^2 - xz \rangle$$

(4) Find the potential function of the following conservative vector field:
$$\mathbf{F} = 2xe^{3y}\,\mathbf{i} + (2 + 3x^2 e^{3y})\,\mathbf{j}$$

Answer:
$$f = x^2 e^{3y} + 2y + C$$

2. Part II

There are 3 problems in Part II, each worth 8 points. On Part II problems show all your work! Your work, as well as the answer, will be graded. Your solution must include enough detail to justify any conclusions you reach in answering the question.

(1) Use Green's theorem to evaluate the line integral

$$\int_C \left(e^{\sqrt{x}} + xy\right) dx + \left(-x^2 + \ln(3 + y^y)\right) dy$$

where C is the border of the square with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Assume the counterclockwise orientation for C.

Solution:

$$
\begin{aligned}
\text{integral} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \iint_D \left(-2x - x\right) dA \\
&= \int_0^1 \int_0^1 -3x \, dx \, dy \\
&= -\frac{3}{2}.
\end{aligned}
$$

(2) Find the work done by the force field $\mathbf{F} = y\mathbf{i} - 2x\mathbf{j} + z\mathbf{k}$ on a particle moving along a line segment from the point $(1, 0, -2)$ to the point $(2, 4, 1)$.

Solution:

The segment is parameterized as $\mathbf{r}(t) = \langle t + 1, 4t, 3t - 2 \rangle$ with $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 1, 4, 3 \rangle$.

Then the vector field is $\mathbf{F}(t) = \langle 4t, -2t - 2, 3t - 2 \rangle$.

The integral is

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(t) \cdot \mathbf{r}'(t) \, dt \\
&= \int_0^1 \left[4t + 4(-2t - 2) + 3(3t - 2) \right] dt \\
&= \int_0^1 (5t - 14) \, dt \\
&= -\frac{23}{2}
\end{aligned}
$$

(3) Evaluate the surface integral $\iint_S \mathbf{F}\, d\mathbf{S}$, where
$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (3 - 2z)\,\mathbf{k}$$
and S is the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the plane $z = 0$, and S has upward orientation.

Solution:

We have $g(z) = 1 - x^2 - y^2$, so $g_x = -2x$ and $g_y = -2y$.

The surface integral is
$$\iint_S \mathbf{F}\, d\mathbf{S} = \iint_D \left[-x(-2x) - y(-2y) + 3 - 2(1 - x^2 - y^2) \right] dA$$
$$= \iint_D (4x^2 + 4y^2 + 1)\, dA$$
$$= \int_0^{2\pi} \int_0^1 (4r^2 + 1)r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (4r^3 + r)\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[r^4 + \tfrac{r^2}{2} \right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{3}{2}\, d\theta$$
$$= 3\pi$$