MA 227 (Calculus-III) Final exam

Show your work. Each problem is 10 points. Tue, Dec 14, 2004

Solve 10 problems for full credit. You can do all 11 problems for extra credit.

1. Find the work done by the force field $\mathbf{F} = y\mathbf{i} - 2x\mathbf{j}$ around the upper semicircle $\{x^2 + y^2 = 9,\ y \geq 0\}$, oriented in the counterclockwise direction.

Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-9\sin^2\theta - 18\cos^2\theta)\, d\theta = -\frac{27\pi}{2}$$

2. (a) Determine whether or not $\mathbf{F} = (2x - y)\mathbf{i} - x\mathbf{j}$ is a conservative vector field.
(b) If it is, find a function f such that $\mathbf{F} = \nabla f$.
(c) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from $(1, 2)$ to $(2, 1)$.

Answers:
(a) Since

$$\frac{\partial P}{\partial y} = -1 \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = -1$$

(they are equal) the field is conservative.

(b) $f = x^2 - xy$.

(c) $f(2, 1) - f(1, 2) = 3$.

3. Use Green's theorem to evaluate the line integral

$$\int_C \left(e^{x^3} - xy\right) dx \, + \left(x^2 + \ln(1 + y^y)\right) dy$$

where C is the square with sides $x = 0$, $x = 1$, $y = 0$, and $y = 1$. Assume the counter-clockwise orientation for C.

Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (2x + x) \, dA = 3/2.$$

4. Use Green's theorem to find the area bounded by the curve

$$\mathbf{r}(t) = (2 - 3\cos t)\mathbf{i} + (5 + 2\sin t)\mathbf{j}, \qquad 0 \le t \le 2\pi$$

Answer: note that the orientation is clockwise! Therefore the standard formulas for the area must be negated:

$$A = -\int_0^{2\pi} (2 - 3\cos t)(2\cos t)\, dt = 6\pi$$

5. Determine whether or not the vector field

$$\mathbf{F} = (2xy - z^2)\mathbf{i} + (x^2 - 2yz)\mathbf{j} - (y^2 + 2xz)\mathbf{k}$$

is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Answer: since $\operatorname{curl} \mathbf{F} = 0$, the field is conservative. Then $f = x^2y - xz^2 - y^2z$.

6. (a) Is there a vector field $\mathbf{G}$ such that

$$\operatorname{curl}\mathbf{G} = 3xy^2\mathbf{i} - y^3\mathbf{j} + (x + yz)\mathbf{k}?$$

Explain.
(b) Write down formulas for the gradient, curl, and divergence, in terms of the ∇ vector. Describe relations between the gradient, curl, and divergence.

Answers:
(a) Since $\operatorname{div}\operatorname{curl}\mathbf{G} = y(\neq 0)$, there is no such $\mathbf{G}$.

(b) $\operatorname{grad} f = \nabla f$, $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$, $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$. For the relations – see the book.

7. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = (y + e^z)\mathbf{i} + (3x - z\sin y)\mathbf{j} - \big(z + \ln(x^2+1)\big)\mathbf{k}$$

and C is a triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$ traversed in this order.

Answer: first of all, we need to compute $\operatorname{curl}\mathbf{F} = 2\mathbf{k}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dA = \int_0^1 \int_0^{1-x} 2\, dy\, dx = 1$$

8. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (x^3 - yz^5)\mathbf{i} + (e^z - 3x^2y)\mathbf{j} - (2z + xy^{-2})\mathbf{k}$$

and S is the sphere $x^2 + y^2 + z^2 = 0.25$ oriented with outward normal vectors.

Answer:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E (-2)\, dV = -2\,\mathrm{Vol}(E) = -\pi/3$$

9. Evaluate the surface integral $\iint_S (x+y+z)\,dS$, where S is the surface with parametric equations $x = u + v$, $y = u - v$, $z = 1 - u$, where $0 \le u \le 1$, $0 \le v \le 1$.

Answer:

$$\iint_D f\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_D (1+u)\sqrt{6}\,dA = \frac{3\sqrt{6}}{2}$$

10. Evaluate the surface integral $\iint_S \mathbf{F}\, d\mathbf{S}$, where

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (5 - 3z)\mathbf{k}$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the plane $z = 0$, and S has upward orientation.

Answer:

$$\begin{aligned}\iint_S \mathbf{F}\, d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \\ &= \iint_D (5x^2 + 5y^2 + 2)\, dA = \int_0^{2\pi} \int_0^1 (5r^2 + 2)\, r\, dr\, d\theta = \frac{9\pi}{2}\end{aligned}$$

11. Evaluate the line integral $\int_C (x + y)\, ds$, where C is a curve consisting of two line segments: one starts at $(-1, -1)$ and terminates at $(3, 0)$, and the other starts at $(3, 0)$ and terminates at $(4, 4)$.

Answer: parametric equations of these line segments are

$$\mathbf{r}_1(t) = \langle 4t - 1, t - 1 \rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \langle t + 3, 4t \rangle$$

(for both, $0 < t < 1$). Note that $|\mathbf{r}_1'| = |\mathbf{r}_2'| = \sqrt{17}$. Thus

$$\int_{C_1} (x + y)\, ds + \int_{C_2} (x + y)\, ds = \int_0^1 \sqrt{17}(5t - 2)\, dt + \int_0^1 \sqrt{17}(5t + 3)\, dt = 6\sqrt{17}$$