Midrange crossing constants of certain graph classes

László A. Székely, University of South Carolina

Graphs mean finite simple graphs on n vertices and m edges. A *drawing* of the graph maps vertices of the graph into distinct points of the plane, and represents edges of the graph with simple curves that connect the two points corresponding to the two vertices of the edge. Curves are not supposed to pass through points corresponding to other vertices than its two endpoints. The *crossing number of a drawing* is computed by counting the number of common interior points for every pair of curves representing edges, and then adding up these numbers for all pairs of curves. The *crossing number* of a graph is the minimum crossing number of its drawings. Let $\kappa(n, m)$ denote the minimum crossing number of graphs on n vertices and m edges.

In 1973, Erdős and Guy conjectured the existence of a real number $C > 0$, such that for any $n \to \infty$ and $m/n \to \infty$, $\lim \kappa(n, m)n^2/m^3 = C$. It was an ambitious conjecture, considering that $\liminf \kappa(n, m)n^2/m^3 > 0$ was showed only ten years later, with the celebrated Crossing Lemma, discovered independently by Ajtai, Chvátal, M. Newborn, and Szemerédi in 1982 and Leighton in 1984. In 2000, Pach, Spencer and Tóth proved the Erdős–Guy conjecture under an additional assumption, $n^2/m \to \infty$. The number C is called the *midrange crossing constant*. (Some additional assumption was needed, as the conjecture fails for $m(n) = \binom{n}{2}$.) Upper and lower bounds have been established for C, but it is not even known whether C is rational or not.

In 2019, Asplund et al. ran into a problem that required the existence of the midrange crossing constant for bipartite graphs, more precisely that the limit above exists if we replace $\kappa(n, m)$ with $\kappa_B(n, m)$, which is the minimum crossing number of *bipartite* graphs on n vertices and m edges. First we thought the limits must be the same, but Czabarka, Reiswig, Zhiyu Wang, and I only could show that the class of bipartite graphs B has its own C_B midrange crossing constant. We do not know if $C = C_B$ or not.

We call a class of graphs $\mathcal{B}$ a PST class, if it contains at least one graph with at least one edge, and the class is closed under taking subgraphs, making vertex disjoint union of, and cloning vertices of members. (Cloning means taking a vertex v, making a copy v' of it, and joining v' to all neighbors of v.) Czabarka, Reiswig, Zhiyu Wang, and I proved that every PST class $\mathcal{B}$ has its own crossing constant, $C_{\mathcal{B}}$. PST classes form an infinite lattice under $\cup$ and $\cap$. In this lattice the maximum element is *the class of all graphs*, the minimum element is *the class B of all bipartite graphs*. If the midrange crossing constants of some two PST classes differ, then the constants C and C_B must also differ.

In 1997, Pach and Tóth claimed $C \leq \frac{16}{27\pi^2} < 0.06005$, which was corrected in 2006 by Pach, Radoičič, Tardos, and Tóth to $C \leq \frac{8}{9\pi^2} < 0.09007$. Their construction is a square grid in the unit square, with negligible random disturbation of every point, and joining two points with a straight line segment, if their distance is less than a threshold. As the calculations are very long, details of them were not published and they are not available. Since 2019, Ackerman has the best constant in the Crossing Lemma, showing $0.0344 < 1/29 \leq C$. A 2017 preprint of Angelini, Bekos, Kaufmann, Pfister, and Ueckerdt essentially showed $0.055 < 16/289 \leq C_B$. These bounds do not separate C and C_B.

In 2020, Czabarka, Singgih, Zhiyu Wang and I gave a new proof to $C \leq \frac{8}{9\pi^2} < 0.09007$ using a spheric construction, with a reasonably simple calculation in spheric geometry.