Stability for the inverse resonance problem for the CMV operator

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I am reporting on joint work with

- Roman Shterenberg (UAB) and
- Maxim Zinchenko (New Mexico).

The CMV operator

Given a sequence $k \mapsto \alpha_k \in \mathbb{D}$ (the unit disk) let $\rho_k = \sqrt{1 - |\alpha_k|^2}$ and

$$U = VW = \begin{pmatrix} 1 & & & & & \\ & -\alpha_2 & \rho_2 & & & \\ & \rho_2 & \overline{\alpha}_2 & & & \\ & & & -\alpha_4 & \rho_4 & \\ & & & \rho_4 & \overline{\alpha}_4 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} -\alpha_1 & \rho_1 & & & \\ \rho_1 & \overline{\alpha}_1 & & & \\ & & -\alpha_3 & \rho_3 & \\ & & \rho_3 & \overline{\alpha}_3 & \\ & & & & \ddots \end{pmatrix}.$$

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- The α_k are called Schur parameters or Verblunsky coefficients.
- U is called a CMV matrix. It is 5-diagonal and maps $\mathbb{C}^{\mathbb{N}_0}$ to itself.
- The restrictions of U, V, and W to $\ell^2(\mathbb{N}_0)$ are unitary operators.

CMV recursion

- For $z \neq 0$ Gesztesy and Zinchenko (2006) introduced invertible 2×2-matrices $T(z,k)$ and the recurrence relation

$$\begin{pmatrix} u(k) \\ v(k) \end{pmatrix} = T(z,k) \begin{pmatrix} u(k-1) \\ v(k-1) \end{pmatrix}, \quad k \in \mathbb{N}. \tag{1}$$

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- Points in the unit disk are connected with those outside since

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- The solutions of (1) with initial conditions $(-1,1)^\top$ and $(1,1)^\top$ are called $\vartheta(z,\cdot)$ and $\varphi(z,\cdot)$, respectively.

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- In particular $Uu = zu$ if $v(z,0) = u(z,0)$.
- Thus, if $(u, v)^\top = \varphi$, we have $v(z,0) = u(z,0) = 1$ and hence $Uu = zu$.

Weyl-Titchmarsh solutions

- Define, for $|z| \neq 1$,

$$u(z,\cdot) = 2z(U-z)^{-1}\delta_0 \in \ell^2(\mathbb{N}_0) \quad \text{and} \quad v(z,\cdot) = \frac{1}{z} W u \in \ell^2(\mathbb{N}_0).$$

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- If $z \neq 0$, then $(u,v)^\perp$ satisfies the CMV recursion and

$$\begin{pmatrix} u \\ v \end{pmatrix}(k) = \vartheta(k) + m(z)\varphi(k) =: \omega(k)$$

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- It follows that

$$m(z) = 1 + u(z,0) = \langle \delta_0, (U+z)(U-z)^{-1}\delta_0 \rangle$$

and hence $m|_{\mathbb{D}}$ is a Caratheodory function ($m(0) = 1$, $\mathrm{Re}(m) > 0$) with representation

$$m(z) = \oint_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$$

where μ is a positive measure on $\partial\mathbb{D}$.

The Fourier transform

- Given a finitely supported $f \in \mathbb{C}^{\mathbb{N}_0}$ let $(\mathcal{F}f)(z) = \sum_{n=0}^{\infty} f(n)\varphi_1(z,n)$ whenever $z \in \partial\mathbb{D}$.

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- In particular, $\langle \varphi_1(\cdot,n), \varphi_1(\cdot,m)\rangle_{L^2(\mu)} = \delta_{n,m}$, i.e., the $\varphi_1(\cdot,n)$ are orthonormal Laurent polynomials which one may also obtain from applying Gram-Schmidt to the sequence $(1, z, 1/z, z^2, 1/z^2, ...)$.

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- The action of U transforms to multiplication by the independent variable, i.e., $\mathcal{F} \circ U \circ \mathcal{F}^{-1} = \mathrm{id}$.

More about the m-function

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- In general m cannot be continued analytically across (all of) the unit circle.
- Even if one can continue, the continuation may be different from the m-function on the other side.

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- The Volterra equation [W., Zinchenko 2010]
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- Either component of $F(\cdot, k)$ is entire of growth order 0.
- Then

$$\nu(z,k) = 2z^{\lceil k/2 \rceil} \Big(\prod_{j=k+1}^{\infty} \rho_j^{-1} \Big) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} F(z,k)$$

satisfies the CMV recursion.

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- m has a meromorphic extension to all of $\mathbb{C}$ (denoted by M).
- ψ_0 cannot have zeros in $\mathbb{D}$. Those outside are called resonances.

Jost solutions III

- For the sequence $\tilde{\nu}(z,k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\nu(1/\overline{z},k)}$ is also a solution of the CMV recursion (and square integrable for $|z| > 1$).

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- An analytic function in the unit disk is (up to an additive constant) determined by its real part on the unit circle.

The inverse resonance problem

Theorem (W., Zinchenko (2010))

The location of the resonances (accounting for multiplicities) determine the Verblunsky coefficients uniquely.

Sketch of proof

- The Verblunsky coefficients are given by the Schur functions (monic orthogonal polynomials) as $\alpha_{k+1} = -\overline{\Phi_k(0)}$.

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- $\psi_0(z) = \psi_0(0) \prod_{k=1}^{\infty}(1 - z/z_k)$ where the z_k are the resonances.
- $|\psi_0(0)|$ is determined since

$$m(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \mathrm{Re}(M(\mathrm{e}^{it}))dt,$$

$m(0) = 1$, and $\mathrm{Re}(M(\mathrm{e}^{it})) = 1/|\psi_0(\mathrm{e}^{it})|^2$. □

Stability

Theorem (Shterenberg, W., Zinchenko (2013))

Suppose α and $\breve{\alpha}$ are two sequences of Verblunsky coefficients with super-exponential decay as before and that $\prod_{j=1}^{\infty}(1-|\alpha_j|) \geq 1/Q$. Assume that the resonances in some ball of radius R, if there are any, are respectively ε-close. Then there is a constant A_0, depending only on γ, η, and Q, such that

$$|\alpha_n - \breve{\alpha}_n| \leq A_0 \left(\varepsilon + \frac{(\log R)^{\gamma/(\gamma-1)}}{R} \right)^{1/\log(6\mathrm{e}Q^2)}$$

for all $n \in \mathbb{N}$.

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- $|\alpha_k - \breve{\alpha}_k| \le |\Phi_{k-1}(0) - \breve{\Phi}_{k-1}(0)| \le \|\Phi_{k-1} - \breve{\Phi}_{k-1}\|_1$ by the mean value theorem ($\|f\|_p^p = \int_{-\pi}^{\pi} |f|^p dt/(2\pi)$).

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- $\|\Phi_{k-1} - \breve{\Phi}_{k-1}\|_1 \le 6Q^2 \|\Phi_{k-2} - \breve{\Phi}_{k-2}\|_1$ by the Schur algorithm.

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- $\Phi_0(z) - \breve{\Phi}_0(z) = \frac{2}{z} \frac{M(z) - \breve{M}(z)}{(1+M(z))(1+\breve{M}(z))}$.

Sketch of proof

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- $\|\Phi_{k-1} - \breve{\Phi}_{k-1}\|_1 \le 6Q^2 \|\Phi_{k-2} - \breve{\Phi}_{k-2}\|_1$ by the Schur algorithm.
- $\Phi_0(z) - \breve{\Phi}_0(z) = \frac{2}{z} \frac{M(z) - \breve{M}(z)}{(1+M(z))(1+\breve{M}(z))}$.
- $|1 + M(z)| \ge \mathrm{Re}(1 + M(z)) \ge 1$.

Sketch of proof

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- If $|\operatorname{Re} f(0)| = |\operatorname{Im} f(0)|$ then $\operatorname{Re} f$ and $\operatorname{Im} f$ have the same 2-norm.

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- If $|\mathrm{Re}\, f(0)| = |\mathrm{Im}\, f(0)|$ then $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ have the same 2-norm.
- We need to estimate $\|\mathrm{Re}\, M - \mathrm{Re}\, \breve{M}\|_2 = \||\psi_0|^{-2} - |\breve{\psi}_0|^{-2}\|_2$.

Sketch of proof

- $|\alpha_k - \check{\alpha}_k| \leq |\Phi_{k-1}(0) - \check{\Phi}_{k-1}(0)| \leq \|\Phi_{k-1} - \check{\Phi}_{k-1}\|_1$ by the mean value theorem ($\|f\|_p^p = \int_{-\pi}^{\pi} |f|^p dt/(2\pi)$).
- $\|\Phi_{k-1} - \check{\Phi}_{k-1}\|_1 \leq 6Q^2 \|\Phi_{k-2} - \check{\Phi}_{k-2}\|_1$ by the Schur algorithm.
- $\Phi_0(z) - \check{\Phi}_0(z) = \frac{2}{z} \frac{M(z) - \check{M}(z)}{(1+M(z))(1+\check{M}(z))}$.
- $|1 + M(z)| \geq \operatorname{Re}(1 + M(z)) \geq 1$.
- If $|\operatorname{Re} f(0)| = |\operatorname{Im} f(0)|$ then $\operatorname{Re} f$ and $\operatorname{Im} f$ have the same 2-norm.
- We need to estimate $\|\operatorname{Re} M - \operatorname{Re} \check{M}\|_2 = \||\psi_0|^{-2} - |\check{\psi}_0|^{-2}\|_2$.
- Hence we need to compare

$$\psi_0(z) = \psi_0(0) \prod_{n=1}^{\infty} (1 - z/z_n) \quad \text{and} \quad \check{\psi}_0(z) = \check{\psi}_0(0) \prod_{n=1}^{\infty} (1 - z/\check{z}_n).$$

Thank you for your attention!