The Spectral Problem for the Dispersionless Camassa-Holm Equation

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AMS meeting in Athens

5. March 2016

I am reporting on joint work with:

- Christer Bennewitz (Lund)
- Malcolm Brown (Cardiff)

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- If $q = 1/4$ or $q = \delta_0$, then $|w|$ has to be a finite measure.
- If $q = x^2/4$ then $|w|$ may be an infinite measure but may not be as large as Lebesgue measure ($\int |w|/(1+|x|) < \infty$)

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- R_0 is compact and the spectrum of T is discrete.

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- Matching constant: $f_+(x, \lambda_n) = \alpha_n f_-(x, \lambda_n)$

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Suppose T and $\check{T}$ have the same spectral data. Then there are continuous functions s and r on $\mathbb{R}$ such that s is bijective, $s' = 1/r^2$, $r > 0$, $r' \in \mathrm{BV}_{\mathrm{loc}}$,

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- Bennewitz, Brown, W. (2014)

Application to Camassa-Holm

- The Camassa-Holm (CH) equation

$$\psi_t - \psi_{txx} - 2\kappa\psi_x + 3\psi\psi_x = 2\psi_x\psi_{xx} + \psi\psi_{xxx},$$

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- CH has many similarities with KdV, in particular, it is the compatibility condition for the linear equations one of which is $-u_{xx} + \frac{1}{4}u = \lambda wu$.

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- $\alpha_n(t) = \mathrm{e}^{t/(2\lambda_n)}\alpha_n(0)$

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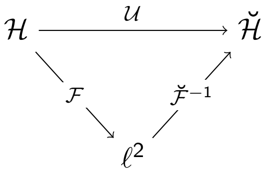

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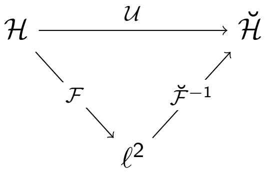

- We need to show that $\mathcal{U}$ defined here is a Liouville transform ($\mathcal{U}^{-1}\breve{u} = r\breve{u} \circ s$):

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- Since, for any x, we have $v(x)\breve{u}(s(x)) = \breve{v}(s(x))u(x)$ we define r by $u(x) = r(x)\breve{u}(s(x))$.

Sketch of proof – Two main ingredients for the details

- The fact that $\mathcal{U} = \breve{\mathcal{F}}^{-1} \circ \mathcal{F} : \mathcal{H} \to \breve{\mathcal{H}}$ is unitary is used to show that S_+ is bounded below and to show that gaps in $\mathrm{supp}(w)$ correspond to those in $\mathrm{supp}(\breve{w})$.

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  - This, in turn, depends on a lemma by De Branges: If F_1 and F_2 are entire and of exponential type 0 and if $\min\{|F_1(z)|, |F_2(z)|\} = o(1)$ uniformly in $\mathrm{Re}(z)$ as $|\mathrm{Im}(z)| \to \infty$, then one of F_1 and F_2 must be identically equal to zero.

Thank you for your attention!